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by

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1. INTRODUCTION

The deterministic linear programming formulation of real world models possesses some inherent unrealistic characteristics. This is due to coefficients in the model which are generally subject to random variations. In the last two decades, linear programs with some coefficients subject to random variations have received considerable attention. These have been studied under different names including probabilistic linear programming, stochastic linear programming, and linear programming under uncertainty. In this study we use the term stochastic linear programming (SLP) to identify a linear programming model with random coefficients.

Although the concept of stochastic linear programming seems to be appealing, the solution of the SLP model raises some serious questions with regard to the computational and theoretical aspects of the model. The specifics of some of these questions will be addressed in later chapters.

Although the primary emphasis of this study is on stochastic linear programming, two other topics have been discussed. Chapter 2 presents a formal methodology for solving deterministic linear programs by the Signal Flow Graph (SFG) method. A procedure to find the inverse of a matrix by utilizing the Mason's gain formula of SFG is also presented in Chapter 2. Chapter 3 reviews the application of the Mellin transform in statistics. The review of literature for these two topics are contained within their respective chapters. Chapter 4 presents an intuitive introduction of stochastic linear programs. In this chapter a new class of stochastic linear program is introduced and some of its properties are described. The review of literature of this topic is given in Sections 1.1 through 1.4. Chapter 5 combines the ideas of Chapters 2, 3, and 4 to present another view of solving stochastic linear programs.

1.1. Literature Review of Stochastic Linear Programming

In the classical linear programming (LP) model:

$$\max Z = cX \tag{1.1-1}$$

subject to (s.t.)

$AX = b$

$X \geq 0$

where:

X = nx1 decision vector

c = 1xn profit vector

b = mx1 resource vector

A = mxn technological coefficient matrix

The parameters in the set (A, b, c) are fixed known numbers, and it is required to determine an optimal decision vector X* subject to the specified constraints. If some or all of the elements in the set (A, b, c) are stochastic, as in real life problems, then classical methods of linear programming will fail to produce a sensible optimal solution. Stochastic linear programming (SLP) is concerned with problems arising when some or all elements of the set (A, b, c) are random variables with known probability density functions. Mandansky (1960) identified two types of stochastic linear programming:

a) "Wait-and-See": In this type of SLP problem one waits till a realization of the random vector θ, where θ denotes an observation of the set (A, b, c), is made and then solves the deterministic LP problem based on the observed random variable θ. By utilizing several observed values of θ, either exactly or approximately, the probability distribution of the maximum value of the objective function and of optimal decision vectors can be derived. Tintner (1955) classified the "Wait-and-See" type of SLP as the "distribution problem."

b) "Here-and-Now": In this type, decisions concerning activity vector X (or on a "strategy" for X) is made in advance or at least without waiting for the realization of random vector θ. The "Here-and-Now" type has also been called the decision problem of SLP.

1.2. The Distribution Problem

The basic approach to solve the distribution problem of SLP is to generate all possible combinations of the basis, and then determine the probability distribution of the objective function. Tintner (1955) was among the first to investigate this type of problem. His basic approach was to take all the possible combinations of the realizations of θ and solve the respective deterministic LP problem. He then used the method of sample moments to fit a probability density function to the obtained values. The shortcomings of this technique is twofold: first, a large number of deterministic LP problems have to be solved and, second, the derived distributions are only an approximation of the actual distribution. Bereanu (1963) considered an SLP where θ is a function of only one random variable. He obtained a closed-form expression for the distribution function of max (Z). Bereanu (1966a, 1966b) also devised a method to determine the distribution of the optimal value of the objective function for the case when the elements of vector c or vector b are random variables. Bereanu assumes that the random variables have finite lower and upper bounds, and he fixes the random variables at their lower bounds. Upon finding the optimal basis associated with the lower bounds of the random variables, he determines the ranges over which the optimal basis remain unchanged using the sensitivity analysis technique of parametric linear programming. He changes the basis and applies the sensitivity analysis to the new bases. This process is repeated until all the optimal basis have been investigated. Then he utilizes the information so obtained to compute the distribution of the optimum value of the objective function.

Prékopa (1966) has given sufficient conditions for the optimal value of the objective function to be normally distributed. Ewbank et al. (1974) propose a method for finding a closed form expression for the cumulative distribution function of the maximum value of the objective function for the cases when elements of vector c or vector b are random variables. The way in which this method differs from previous ones is only in the procedure for determining the probability of a basis being optimal.

Bereanu (1976) gives a sufficient condition that the optimal value of a linear program be a continuous function of its coefficients, and proves a necessary and sufficient condition that an SLP has optimal value.

1.3. The Decision Problem

The general approach to solve this type of problem has been to select some criterion and then solve the equivalent deterministic program. These equivalent programs are normally convex programs which in general are nonlinear. Dantzig (1955) was among the first to introduce this type of problem and he named it "linear programming under uncertainty." In the last two decades several classes of decision problems have evolved. Figure 1-1 depicts the major classes of decision problems.

1.3.1. Active approach

Tintner (1960) and Sengupta et al. (1963a,b) developed the active approach of solving SLP. The basic idea behind this approach is to introduce additional decision variables defined by the resource allocation matrix $D = [d_{ij}]$ $(i = 1,\ldots,m;\ j = 1,\ldots,n)$. The model of (1.1-1) is modified as follows:

$$\max Z = cX$$

$$\text{s.t. } AX \leq bD$$

$$X \geq 0$$

where (A, b, c) is a random vector

$$0 \leq d_{ij} \leq 1$$

$$\sum_{j=i}^{n} d_{ij} = 1$$

The decision-maker has to decide on a fixed value of d_{ij} to maximize the objective function based on a specific criterion. Of several possible criteria one is to maximize the expected value of the objective function.

1.3.2. Stochastic programs with recourse

Dantzig's (1955) model "linear programming under uncertainty" assumes that the elements of vector b of model (1.1-1) are random variables with known distribution functions. The model considered is that of finding the optimum value of vector X in the following model.

$$\begin{aligned} Z = \min_{X} E[CX + \min_{y} gy] \\ \text{s.t. } AX = b \\ TX + WY = P \qquad\qquad (1.3\text{-}1) \\ X \geq 0, \quad y \geq 0 \end{aligned}$$

In the above model the random parameter space of each b_i has been divided into two disjoint classes, one satisfying the constraints and the other not satisfying the constraints. If the latter is nonempty, then with a finite probability the ith constraint is violated. For the first stage of the problem the values of X and b are assumed to be known, and supposing a penalty cost (g_i) is known for each ith constraint violation then the mean total penalty cost (gy) is minimized. If we denote this minimum by Q(X,b), then the objective function of second stage becomes

$$Z = \min E[CX + Q(X)] \qquad X \geq 0$$

where

$$Q(X) = EQ(X,b)$$

Note in the two stage formulation of (1.3-1) the selection of vector X is optimal if it leads to the minimum of expected cost including penalty cost of y. Thus the central emphasis is on the problem of finding an optimal X given the penalty costs of constraint violation and the sequential observations of the random elements of b. Walkup and Wets (1967) studied the natural extension of Dantzig (1955) two-stage model to the more general case. In model (1.3-1) Walkup and Wetts (1967) assumed that not only b is a random variable, but also C, g, T and W as well as P are random variables. They coined the name of "stochastic programs with recourse" for their proposed model. We note that two stage programming is a special case of stochastic programs with recourse. Wets (1972) has explored the generalization to more than two stages under the assumption that the random variables in any stage are independent of the random variables in the preceding stages.

1.4. Chance Constrained Programming

Chance constrained programming (CCP) can best be described as an attempt to optimally allocate resources in situations where Triplet (A, b, c) is random, and the decision-maker requires one or more constraints (including the objective function) to be satisfied. However, not all constraints may be satisfied every time. Charnes and Cooper (1959) pioneered this concept, and during almost the last two decades their original idea has been expanded and strengthened.

A chance constraint admits as many interpretations as does the probability operator, e.g., total or conditional, and also the decision-maker might employ different functionals such as minimization of expected cost, maximization of the probability of some event. Hence CCP is a flexible tool, and the choice of the suitable and analyzable model of a particular situation rests upon the management scientists. The basic idea behind the solution procedures of CCP is to solve the equivalent deterministic program which in almost all cases are nonlinear programs.

The basic model proposed by Charnes and Cooper (1959) assumes that only vector b has random variation. The model of (1.1-1) with chance constraint becomes

$$\max Z = cX$$
$$\text{s.t.} \quad P\{AX \leq b\} \geq 1-\alpha$$
$$X \geq 0$$

where the ith constraint

$$P\{\sum_{j=1}^{n} a_{ij}x_j \leq b_i\} \geq 1-\alpha \quad (i = 1,\ldots,m)$$

for instance, is realized with a minimum probability of $1-\alpha_i$ $(0 < \alpha_i < 1)$. The case of joint chance-constraints in CCP (i.e., where the restriction is on the joint probability of a multivariate random event) has been investigated by Miller and Wagner (1965) and by Prékopa (1970). The paper by Prékopa (1970) investigates the regions in the multivariate normal space where the transformed problem remains a concave program and develops an algorithm based on feasible direction methods. Some of the most widely used functionals are the ones introduced by Charnes and Cooper (1963). They included: a) the "E-model" where the objective function is max E [CX]; b) the "V-model" where the objective is to minimize the variance of the objective function, i.e., $\min E[CX-C_0X_0]^2$; c) the "P-model" where it is desired to maximize the probability β that CX does not exceed a given constant, e.g., C_0X_0, i.e., $\max P\{C_0X_0 \geq CX\} \geq \beta$. The CCP employs a preassigned class of admissable stochastic decision rules which represent the operational prescriptions of the model.

Eisner, Kaplan, and Soden (1971) discuss the admissible decision rules for the E-model. Also Garstka and Wets (1974) present a survey of decision rules in stochastic programming.

Mandansky (1960) and Mangasarian (1964) by use of inequalities have demonstrated the relationship between some widely used functionals. These results could be useful in establishing some upper or lower bound on the objective function although these bounds might not be very sharp.

It should be mentioned that CCP has the basic difficulty that the statistical distribution of the objective function becomes very complicated if random variables are not distributed normally. Hence most practical situations reported in the literature possess this assumption.

Figure 1-1 portrays the major classes of stochastic linear programming and their relationship to each other.

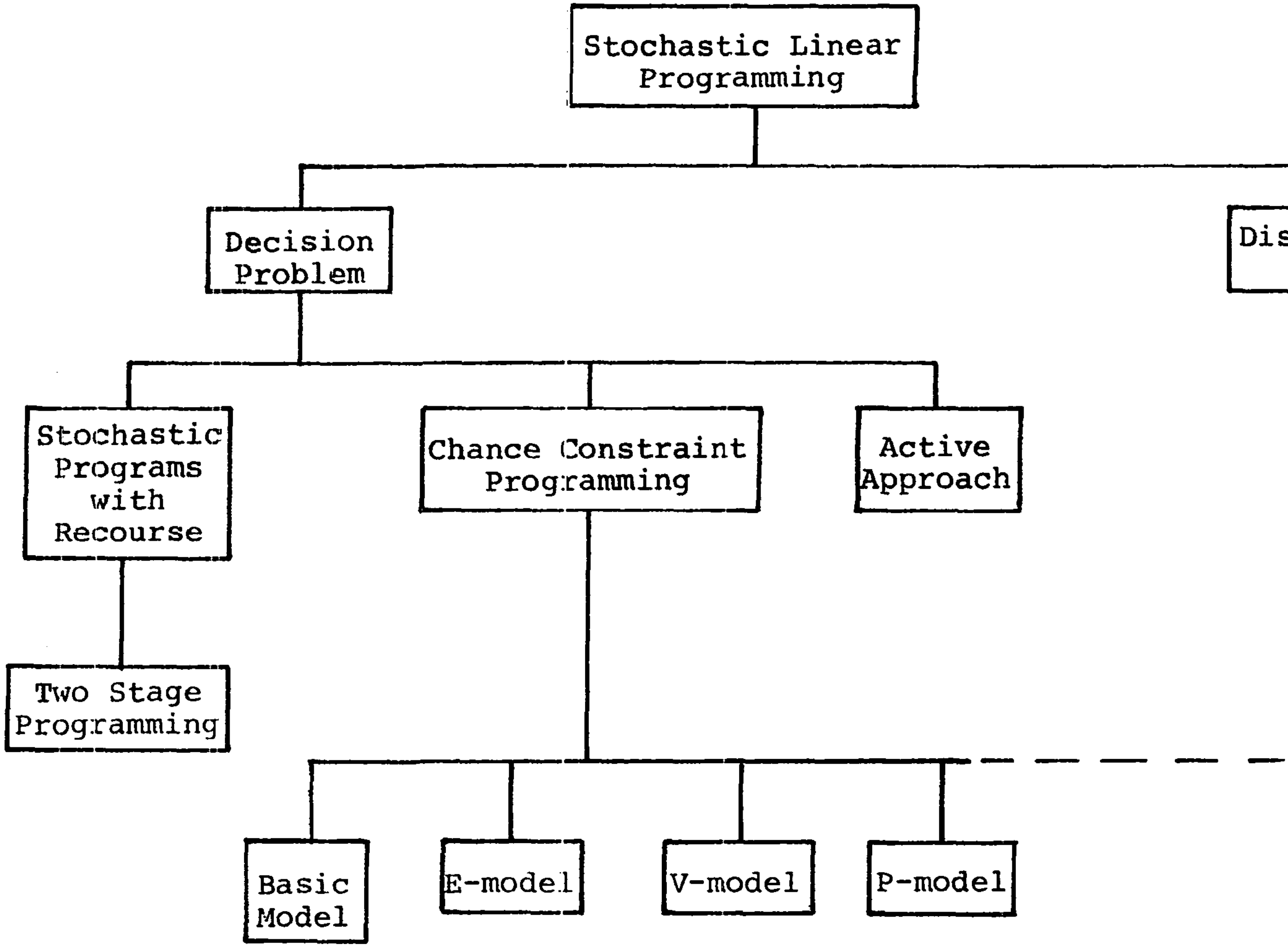


Figure 1-1. Major classes of stochastic linear programming

2. SIGNAL FLOW GRAPH SOLUTION FOR LINEAR PROGRAMMING

2.1. Introduction

Linear programming deals with the maximization or minimization of a linear function, called an objective function, in the presence of a set of linear equations called constraints. Without loss of generality a linear program can be represented as

Maximize $Z = CX$

Subject to: $AX = b$

where:

> X is an nxl vector of decision variables; C is an lxn vector of profit coefficients; A is an mxn matrix of technological coefficients; and b is an mxl vector of resources.

The solution to the above problem is $X_B = B^{-1}b$ where X_B is the vector of basic decision variables and B^{-1} is the inverse of matrix B, a submatrix of A associated with the basic variables. In this chapter we present a graphical method to solve the LP problem using signal flow graph (SFG) (for a review of SFG see Appendix C). Tonomura (1972) was the first to introduce without proof the basic application of SFG in linear optimization, the work presented in this chapter is based on his work with some modification and extensions.

2.2. Finding the Inverse of a Matrix by SFG

Consider a square matrix $A = [a_{ij}]$ $(i = 1, m;\ j=1, m)$, from matrix theory it is known that A^{-1} exists if and only if $|A| \neq 0$. To determine the inverse of matrix A we propose the following method.

1. Augment the matrix by a unit vector i.e., $(A|1)$ where 1 is an mxl vector of positive ones.
2. Normalize the augmented matrix $(A|1)$ by dividing all elements in row i by a_{ii} $(i = 1,m)$. Division by zero is not allowed. So if a_{ii} of row i $(i = 1,m)$ is equal to zero, interchange the columns of matrix A such that the elements of the main diagonal are not equal to zero. If such a matrix can not be found the rank of the matrix A must be less than m and the inverse does not exist. Let B denote the normalized augmented matrix which can be represented by its columns as $(b_1, b_2, \ldots, b_m | b_{m+1})$ where $b_j = \frac{a_{ij}}{a_{ii}}$ $(i = 1,m;\ j=1,m)$ and $b_{m+1} = \frac{1}{a_{ii}}$ $(i=1,m)$. Let us call the b_j $(j=1,m)$ the base columns, and b_{m+1} the augmented column.
3. Construct a graph consisting of 2m nodes. m nodes correspond to base columns b_j $(j=1,m)$ and m nodes correspond to the elements of augmented

column, b_{m+1}. This graph can have a maximum of mm-m+m = mm branches. The transmittance (transmittance $T_{ji} = \frac{y_i}{x_j}$ represents the linear dependency between a dependent variable y_i and an independent variable x_j. T_{ji} is the gain from node x_j to node y_i. See Appendix C) of the branches between the jth base column node and other column nodes are the negative elements of the jth row of matrix $B = [b_{ij}]$ ($i = 1,m$; $j=1,m$, $i \neq j$), and these branches emanate from the column nodes b_i ($i=1,m$, $i \neq j$). For each column node b_j there exists a corresponding augmented node $b_{j,m+1}$, and these two nodes are connected with a branch from the augmented node to the column node. The transmittance of each branch is equal to the jth element of the column vector $b_{m+1} = \frac{1}{a_{ii}}$ ($i=1,m$) as shown in Figure 2-1.

4. Using Mason's gain formula of SFG (see Appendix C) calculate the transmittance from the augmented column nodes $b_{j,m+1}$ ($j=1,m$) to the base column node b_i ($i=1,m$) i.e., $T_{b_{j,m+1} \rightarrow b_i}$ ($i=1,m$, and for all j). The vector of $T_{b_{j,m+1} \rightarrow b_i} = T_{ij}$ ($i=1,m$) will be the ith row of the matrix A^{-1}.

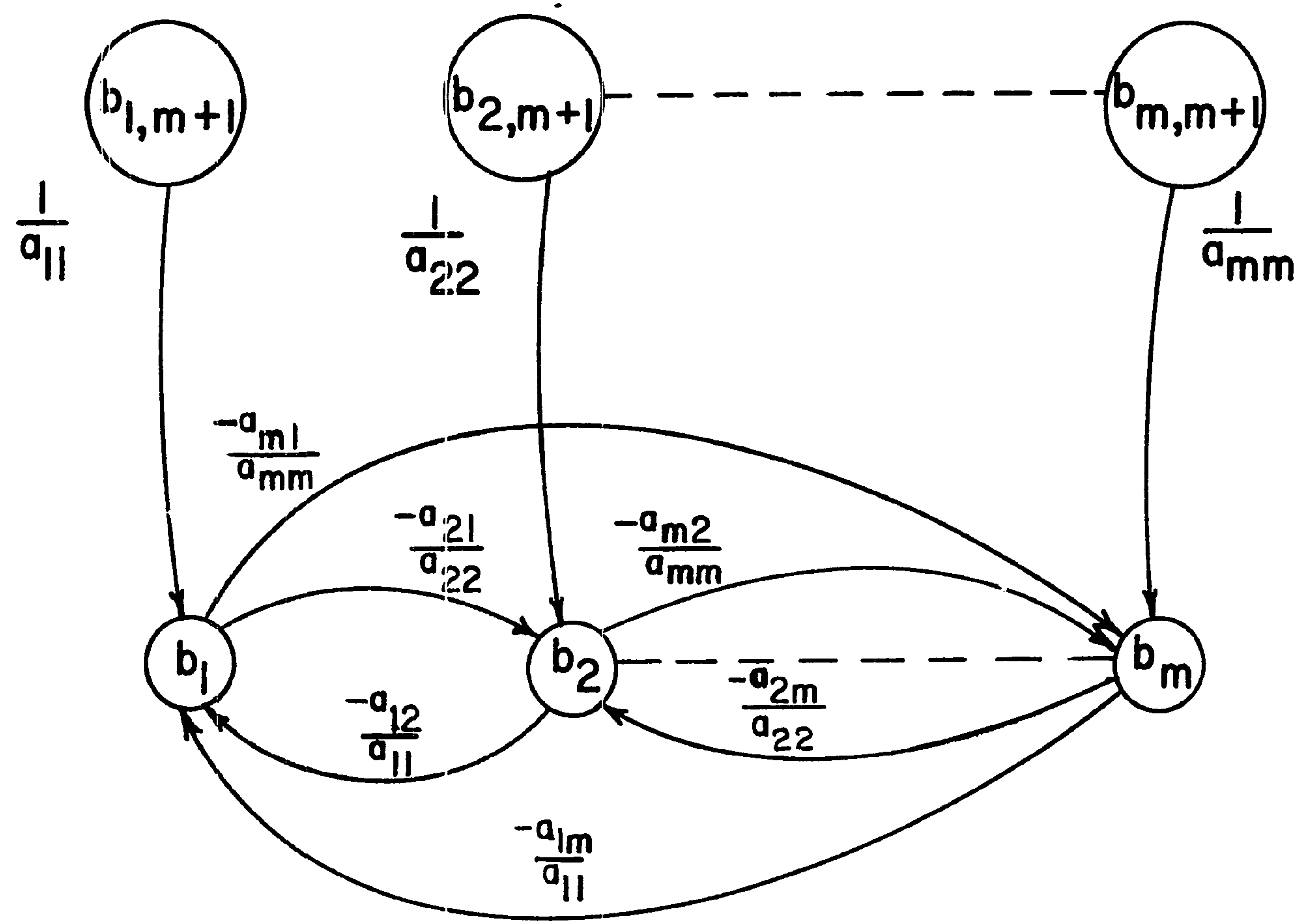


Figure 2-1. Signal flow graph representation of a matrix

Example 2.2-1 illustrates the proposed solution method of finding the inverse of a matrix by SFG.

Example 2.2-1:

Given matrix $A = [a_{ij}]$ $(i=1,2,\ j=1,2)$ find A^{-1}.

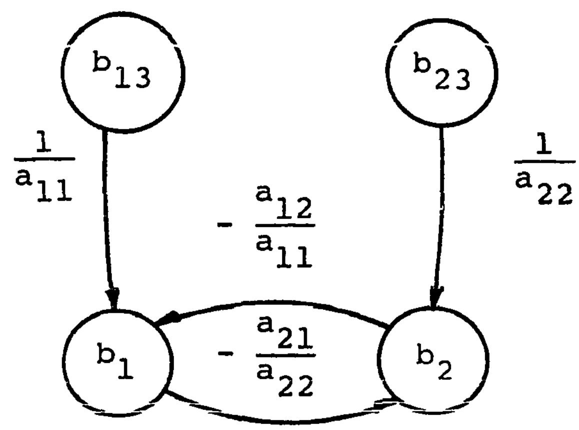
Step 1:

$$(A|1) = \left(\begin{array}{cc|c} a_{11} & a_{12} & 1 \\ a_{21} & a_{22} & 1 \end{array}\right)$$

Step 2:

$$B = \left(\begin{array}{cc|c} 1 & \dfrac{a_{12}}{a_{11}} & \dfrac{1}{a_{11}} \\ \dfrac{a_{21}}{a_{22}} & 1 & \dfrac{1}{a_{22}} \end{array}\right) = (b_1, b_2 | b_3)$$

Step 3:



Step 4:

$$T_{b_{13}\to b_1} = T_{11} = \frac{\frac{1}{a_{11}}}{1 - \frac{a_{12}a_{21}}{a_{11}a_{22}}} = \frac{a_{22}}{a_{11}a_{22}-a_{12}a_{21}}$$

$$T_{b_{23}\to b_1} = T_{12} = \frac{-\frac{a_{12}}{a_{11}a_{22}}}{1 - \frac{a_{12}a_{21}}{a_{11}a_{22}}} = \frac{-a_{12}}{a_{11}a_{22}-a_{12}a_{21}}$$

$$T_{b_{13}\to b_2} = T_{21} = \frac{-\frac{a_{21}}{a_{11}a_{22}}}{1 - \frac{a_{12}a_{21}}{a_{11}a_{22}}} = \frac{-a_{21}}{a_{11}a_{22}-a_{12}a_{21}}$$

$$T_{b_{23}\ b_2} = T_{22} = \frac{\frac{1}{a_{22}}}{1 - \frac{a_{12}a_{21}}{a_{11}a_{22}}} = \frac{a_{11}}{a_{11}a_{22}-a_{12}a_{21}}$$

Due to the equivalence between Cramer's rule of solving a system of equations and Mason's formula (see Appendix C), the SFG method of finding an inverse of a matrix is similar to the adjoint method of finding an inverse. However, the SFG method can be superior to the adjoint method where the matrix A has a high degree of sparsity.

2.3. Improving a Basic Feasible Solution by SFG Method

Regardless of the method of solution used, the basic results of theorems of linear programming remain unchanged. But the way one arrives at these conclusions is a function of the method used. In this section we develop the conditions of improving a basic feasible solution which in concept is identical to the simplex algorithm, but due to the properties of SFG it differs in its appearance.

Let us consider the following linear programming problem.

Problem I:

$$\begin{aligned} &\text{maximize (max) } Z = C\,X \\ &\text{subject to (s.t.) } AX = b \\ &\qquad\qquad\qquad\quad X \geq 0 \end{aligned}$$

Where A is a mxn matrix with rank m; C is a 1xn vector, and b is an mx1 vector.

Suppose that there exists an arbitrary feasible solution $X = \binom{X_B}{X_N}$ to the Problem I, then $X_B \geq 0$ and $X_N \geq 0$. Thus Problem I can be written as follows:

$$Z = (C_B\ C_N)\ \binom{X_B}{X_N}$$

$$\text{s.t.:}\quad (B,N)\ \binom{X_B}{X_N} = b$$

$$X_B \geq 0,\ X_N \geq 0$$

or

$$Z = C_B X_B + C_N X_N \quad (2.3\text{-}1)$$

$$\text{s.t.: } BX_B + NX_N = b \quad (2.3\text{-}2)$$

Multiplying Equation (2.3-2) by B^{-1} on the right hand side, and rearranging obtains

$$X_B = B^{-1}b - B^{-1}N\ X_N. \quad (2.3\text{-}3)$$

Let

$$T_N = -B^{-1}N \quad \text{and} \quad T_B = B^{-1}.$$

Substituting the above in Equation (2.3-3), the basic solution becomes

$$X_B = T_B b + T_N X_N \quad (2.3\text{-}4)$$

and the optimal value can be written as

$$Z = C_B T_B b + C_B T_N X_N + C_N X_N$$

Let

$$Z_0 = C_B T_B b$$

then

$$Z = Z_0 + (C_B T_N + C_N) X_N$$

and denoting $C_B T_N + C_N$ by T_{NZ}

$$Z = Z_0 + T_{NZ}X_N = Z_0 + \sum_{j\varepsilon K} (T_{NZ})_j X_j \qquad (2.3\text{-}5)$$

where

$$(T_{NZ})_j = C_B)T_N)_j + C_j$$

and $K = \{j|X_j$ is nonbasic$\}$. Since our objective is to maximize Z, it is to our advantage to increase the X_j whenever $(T_{NZ})_j$ is positive [i.e., $C_B(T_N)_j+C_j>0$]. The greatest increase in Z will occur if the X_j which has the largest value of $(T_{NZ})_j$ is selected.

As X_j is increased, from its present level of zero, the current basic variables must be modified. Hence,

$$X_B = T_B b + (T_N)_j X_j \qquad (2.3\text{-}6)$$

where $(T_N)_j$ is the jth column of matrix T_N associated with the nonbasic variable X_j. Denoting the components of $T_B b$ and $(T_N)_j$ by $\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_n$ and $T_{N1}, T_{N2}, \ldots T_{Nm}$ respectively, the Equation (2.3-6) is shown as follows:

$$\begin{bmatrix} X_{B1} \\ X_{B2} \\ \vdots \\ X_{Br} \\ \vdots \\ X_{Bm} \end{bmatrix} = \begin{bmatrix} \bar{b}_1 \\ \bar{b}_2 \\ \vdots \\ \bar{b}_r \\ \vdots \\ \bar{b}_m \end{bmatrix} + \begin{bmatrix} T_{N1} \\ T_{N2} \\ \vdots \\ T_{Nr} \\ \vdots \\ T_{Nm} \end{bmatrix} X_j \qquad (2.3\text{-}7)$$

If $T_{Nr} \geq 0$, then X_{Br} increases as X_j increases, thus X_{Br} continues to be nonnegative and X_j can increase without bound. If $T_{Nr} < 0$, then X_{Br} will decrease as X_j increases.

In order to satisfy the nonnegativity condition, X_j is increased until the first basic variable X_{Br} reaches zero. Further examination of Equation (2.3-7) reveals that the first basic variable reaching zero corresponds to the maximum of $\bar{b}_r/T_{Nr}$ for negative T_{Nr}. More precisely,

$$\frac{\bar{b}_j}{T_{Nj}} = \underset{1\leq r\leq m}{\text{maximum}} \left\{\frac{\bar{b}_r}{T_{Nr}}:\ T_{Nr} < 0\right\} = -X_j. \tag{2.3-8}$$

In the absence of degeneracy (i.e., $\bar{b}_j > 0$) $\frac{\bar{b}_j}{T_{Nj}} < 0$, and hence $X_j = -\frac{\bar{b}_j}{T_{Nj}}$.

From Equation (2.3-5) and the fact that $(T_{NB})_j = C_B(T_N)_j + C_j > 0$, it follows that $Z>Z_0$ and the objective function strictly increases.

The commonly used simplex algorithm moves also to a better feasible solution after each iteration. The simplex algorithm accomplishes this by changing the value of one judiciously selected nonbasic variable from its present value of zero to some nonnegative value such that the objective value is increased the most. Like SFG method, the simplex algorithm tries to change the value of only one nonbasic variable at each iteration. The present basic variable that leaves the basis is selected in such a manner that the feasibility of the new basic variables is assured (for a derivation of simplex algorithm, see any linear programming textbook; e.g. Randolph and Meeks (1978)). From the foregone discussion the similarities between SFG method and simplex algorithm are evident.

2.4. SFG Procedures of Simplex Method

To solve the LP Problem I, as defined in Section 2.3, we need to establish some graphical conventions and also recast some of the terminologies of the simplex method into the SFG parlance. Table 2-1 defines the graphical symbols used in SFG procedure.

Table 2-1. Graphical symbols of SFG LP

Symbol	Description
b	Resource or supply node
$\boxed{X_B}$	Basic variable node
X_N	Nonbasic variable node
Z	Objective variable node
$\rightarrow$	Interrelationship between the above nodes

In order to obtain the SFG representation of LP Problem I, where the number of decision variables exceeds the number of equations i.e., m<n, we must decide upon which m elements of decision vector X will form the basic variables vector X_B.

When this decision has been made Problem I can be written as follows

$$\max \quad Z = (C_B | C_N)\binom{X_B}{X_N}$$

$$\text{s.t.} \quad (B|N)\binom{X_B}{X_N} = b$$

$$X_B,\ X_N \geq 0$$

Using the SFG terminology Z and X_B are the dependent variables (see Appendix C). Putting the above LP model in SFG standard form we obtain

$$\max \quad Z = C_B X_B + C_N X_N$$

$$\text{s.t.} \quad X_B = -B^{-1} N X_N + B^{-1} b \qquad (2.4\text{-}1)$$

$$X_B \geq 0,\ X_N \geq 0$$

Using the symbols of Table 2-1 the SFG of Equation 2.4-1 is depicted by Figure 2-3.

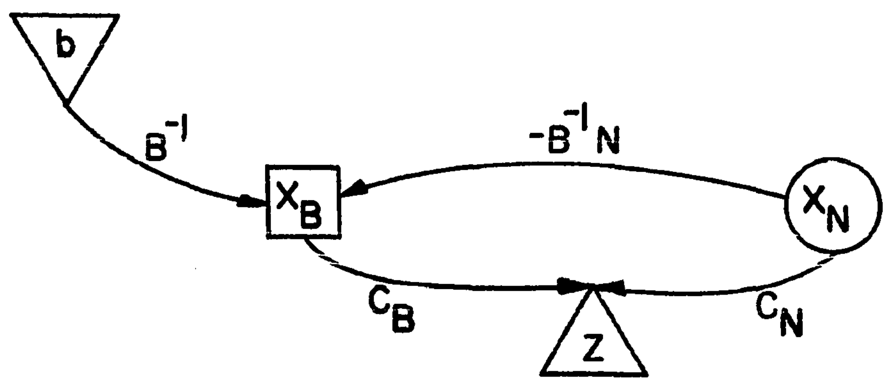


Figure 2-3. SFG representation of Equation 2.4-1

The value of dependent and independent variables can be found by determining the transmittance from the source nodes to the desired variable and then multiplying by the value of the source node.

$$X_B = T_{b \to X_B} b = B^{-1} b$$

$$Z = T_{b \to Z} b = C_B B^{-1} b$$

$$X_N = T_{b \to X_N} = 0$$

where:

$$T_{b \to X_B} \equiv T_B = [T_{ij}] \quad (i=1,m;\ j=1,m)$$

The element T_{ij} is the transmittance from resource j to basic variable i.

$$T_{b \to Z} = [T_{Zj}] \quad (j=1,m)$$

The element T_{Zj} is the transmittance from resource j to the objective variable Z.

2.4.1. Decision rules of SFG procedure

1. Select the nonbasic variable X_i $(i=n-m,n)$ to enter the basis such that its transmittance to Z has the largest positive value. In other words, select that nonbasic variable which contributes the most to the value of the objective function:

$$\max_{i\varepsilon K} [T_{X_i \to Z} | T_{X_i \to Z} > 0] = T_{X_k \to Z} \quad k\varepsilon K \tag{2.4-2}$$

where

$K = \{i | X_i$ is nonbasic$\}$.

Equation 2.4-2 implies that X_k $k\varepsilon K$ is a candidate to enter the basis.

2. Select the basic variable X_j ($j=1,m$) to leave the basis such that the ratio of the current value of X_j (i.e., $T_{b \to X_j}$) to the transmittance from the candidate variable X_k to X_j (i.e. $T_{X_k \to X_j}$) is the largest negative value or mathematically:

Let $R_j = T_{b \to X_j} / T_{X_k \to X_j}$

then

$$\max_{j\varepsilon m} [R_j | R_j < 0] = X_l \quad l\varepsilon m \tag{2.4-3}$$

where

$m = \{j | X_j$ is basic$\}$ and $T_{X_k \to X_j}$ is the kjth element of matrix T_N. Equation 2.4-3 implies that X_l, $l\varepsilon m$ is a candidate to leave the basis as was demonstrated in Section 2.3.

3. A current basic feasible solution is optimal if $T_{X_i \to Z} \leq 0$ $\forall$ $i\varepsilon K$. In the following example, 2.4-1, the SFG procedure of solving a LP problem is illustrated.

Example 2.4-1:

$$\max Z = 5X_1 + 4X_2$$

$$\text{s.t.} \quad X_1 + 3X_2 + X_3 = 6$$

$$2X_1 - X_2 + X_4 = 4$$

$$X_1, X_2, X_3, X_4 \geq 0$$

Stage 1:

$$\max Z = 5X_1 + 4X_2$$

$$\text{s.t.} \quad X_3 = -X_1 - 3X_1 - 3X_2 + 6$$

$$X_4 = -2X_1 + X_2 + 4$$

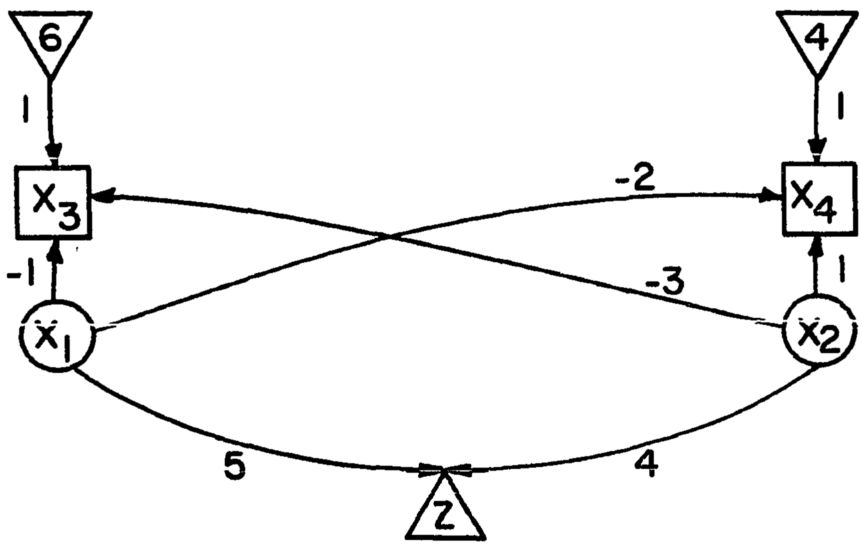


Figure 2-4. SFG representation of stage 1 of Example 2.4-1

$$X_1 = (T_{6\to X_1})(6) + (T_{4\to X_1})(4) = 0$$

$$X_2 = (T_{6\to X_2})(6) + (T_{4\to X_2})(4) = 0$$

$$X_3 = (T_{6\to X_3})(6) + (T_{4\to X_3})(4) = (1)(6) + 0 = 6$$

$$X_4 = (T_{6\to X_4})(6) + (T_{4\to X_4})(4) = 0 + (1)(4) = 4$$

$$Z = (T_{6\to Z})(6) + (T_{4\to Z})(4) = 0$$

Determination of entering and leaving variables:

$$T_{X_1\to Z} = 5$$

$$T_{X_2\to Z} = 4$$

Since

$\text{Max}[T_{X_1\to Z}, T_{X_2\to Z}] = 5$, X_1 is a candidate to enter the basis.

$$T_{X_1\to X_3} = -1$$

$$R_3 = \frac{6}{-1} = -6$$

$$T_{X_1\to X_4} = -2$$

$$R_4 = \frac{4}{-2} = -2$$

Note that

$\text{Max}[R_3, R_4] = -2$ so X_4 is a candidate to leave the basis.

Stage 2:

$$\max Z = 5X_1 + 4X_2$$

$$\text{s.t.} \quad X_3 = -X_1 - 3X_2 + 6$$
$$X_1 = \frac{1}{2} X_2 - \frac{1}{2} X_4 + \frac{1}{2}(4)$$

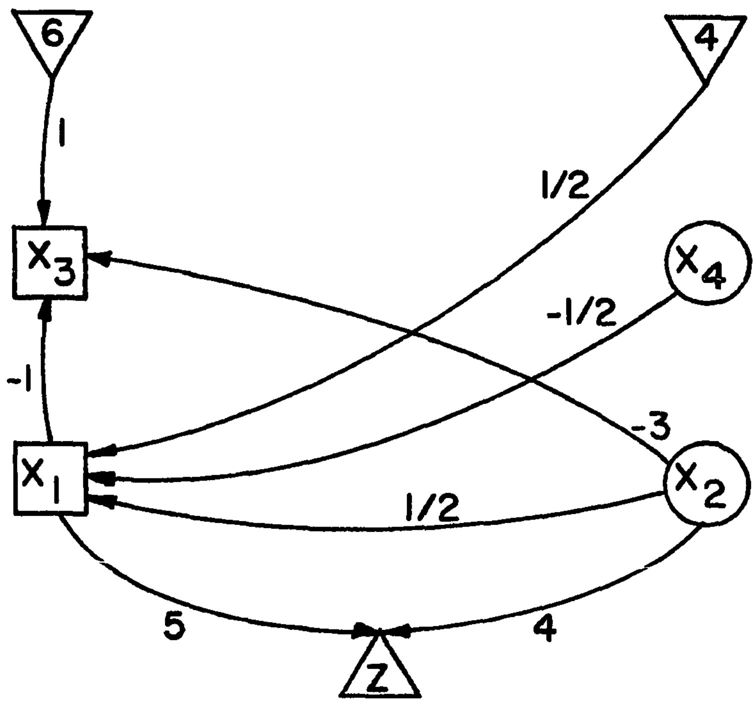


Figure 2-5. SFG representation of Stage 2 of Example 2.4-1

$$X_1 = (T_{6\to X_1})(6) + (T_{4\to X_1})(4) = 0 + (\tfrac{1}{2})(4) = 2$$

$$X_2 = (T_{6\to X_2})(6) + (T_{4\to X_2})(4) = 0$$

$$X_3 = (T_{6\to X_3})(6) + (T_{4\to X_3})(4) = (1)(6) + (\tfrac{-1}{2})(4) = 4$$

$$X_4 = (T_{6\to X_4})(6) + (T_{4\to X_4})(4) = 0$$

$$Z = (T_{6\to Z})(6) + (T_{4\to Z})(4) = 0 + (\tfrac{5}{2})(4) = 10$$

Determination of entering and leaving variables:

$$T_{X_2 \to Z} = 4$$

$$T_{X_4 \to Z} = \frac{-5}{2}$$

Therefore, X_2 is a candidate to enter the basis.

$$T_{X_2 \to X_1} = \frac{1}{2}$$

$$R_1 = \frac{2}{\frac{1}{2}} = 4$$

$$T_{X_2 \to X_3} = -3 - \frac{1}{2} = \frac{-7}{2}$$

$$R_3 = \frac{4}{-\frac{7}{2}} = \frac{-8}{7}$$

Therefore, X_3 is a candidate to leave the basis.

Stage 3:

$$\max Z = 5X_1 + 4X_2$$

$$\text{s.t.} \quad X_2 = -\frac{1}{3} X_1 - \frac{1}{3} X_3 + \frac{1}{3}(6)$$

$$X_1 = \frac{1}{2} X_2 - \frac{1}{2} X_4 + \frac{1}{2}(4)$$

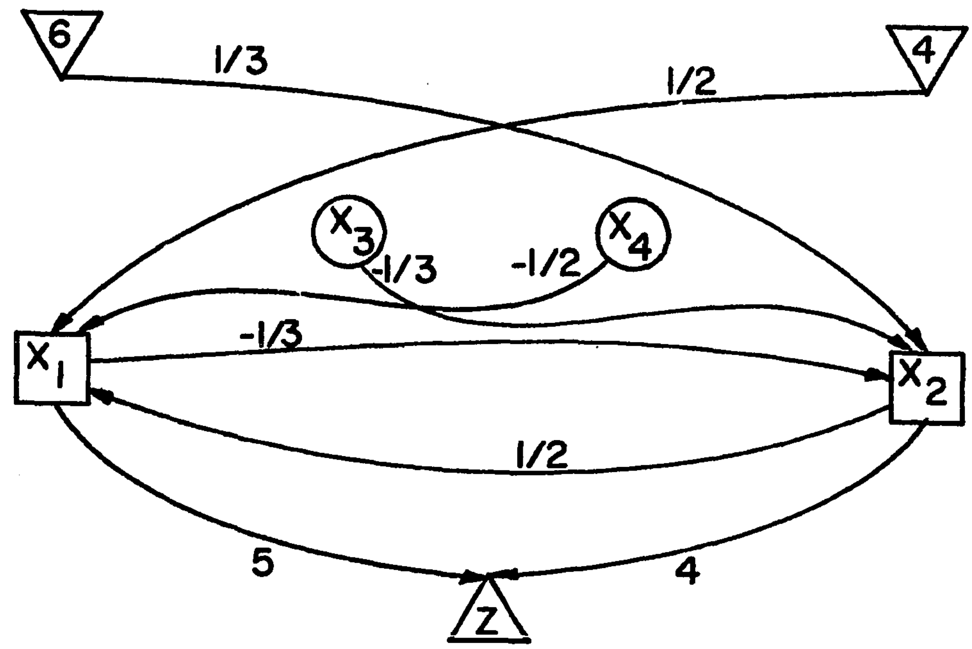


Figure 2-6. SFG representation of Stage 3 of Example 2.4-1

$$X_1 = (T_{6\to X_1})(6) + (T_{4\to X_1})(4) = \left(\frac{\frac{1}{6}}{1+\frac{1}{6}}\right)(6) + \left(\frac{\frac{1}{2}}{1+\frac{1}{6}}\right)(4) = \frac{18}{7}$$

$$X_2 = (T_{6\to X_2})(6) + (T_{4\to X_2})(4) = \left(\frac{\frac{1}{3}}{1+\frac{1}{6}}\right)(6) + \left(\frac{-\frac{1}{6}}{1+\frac{1}{6}}\right)(4) = \frac{8}{7}$$

$$X_3 = (T_{6\to X_3})(6) + (T_{4\to X_3})(4) = 0$$

$$X_4 = (T_{6\to X_4})(6) + (T_{4\to X_3})(4) = 0$$

$$Z = (T_{6\to Z})(6) + (T_{4\to Z})(4)$$

$$Z = \left(\frac{(\frac{1}{3})(4) + (\frac{1}{3})(\frac{1}{2})(5)}{1 + \frac{1}{6}}\right)(6)$$

$$+ \left(\frac{(\frac{1}{2})(5) + (\frac{1}{2})(-\frac{1}{3})(4)}{1 + \frac{1}{4}}\right)(4)$$

$$= (\frac{13}{7})(6) + (\frac{11}{7})(4) = \frac{122}{7}$$

Determination of entering and leaving variables.

$$T_{X_3 \to Z} = \frac{(-\frac{1}{3})(4)}{1 + \frac{1}{6}} + \frac{(-\frac{1}{3})(\frac{1}{2})(6)}{1 + \frac{1}{6}} = \frac{-13}{7}$$

$$T_{X_4 \to Z} = \frac{(-\frac{1}{2})(5)}{1 + \frac{1}{6}} + \frac{(-\frac{1}{2})(-\frac{1}{3})(4)}{1 + \frac{1}{6}} = -\frac{11}{7}$$

Since $T_{X_3 \to Z}$ and $T_{X_4 \to Z}$ are nonpositive the present basic solution $X_1 = \frac{14}{5}$ and $X_2 = \frac{8}{5}$ is optimal.

The basic equations of solving LP problems by SFG procedure is summarized below

$$X_j = \sum_{i=1}^{m} (T_{b_i \to X_j}) b_i \quad \forall\ j \tag{2.4-4}$$

$$Z = \sum_{i=1}^{m} (T_{b_i \to Z}) b_i \tag{2.4-5}$$

$$\overline{C}_l = C_l - Z_l = T_{X_l \to Z} \quad \forall\ l \varepsilon n \tag{2.4-6}$$

$$(B^{-1})_j = [T_{b_i \to X_j}] \quad (i=1,m) \text{ and } \forall (j \mid X_j \text{ is basic}) \tag{2.4-7}$$

where $(B^{-1})_j$ is the jth column of B^{-1}. An interesting observation can be made.

Recall that dual variables y_i, $(i=1,m)$, are defined as $y_i = \frac{\partial Z^*}{\partial b_i}$, where Z^* is the optimal value of Z. Assuming an optimal solution has been determined. Then differentiating Equation 2.4-5 with respect to b_i obtains

$$\frac{\partial Z^*}{\partial b_i} = y_i = T_{b_i \to Z} \quad (i=1,m) \tag{2.4-8}$$

On the other hand $y = C_B B^{-1}$, and the negative of the dual variables can be observed from the $\bar{C}$-row ($\bar{C} = C_j - C_B B^{-1} P_j$) of the final tableau of the simplex method under the columns of the starting solution. Thus from Equation 2.4-6 the dual variables are:

$$y_i = -T_{X_i \to Z} \tag{2.4-9}$$

where i is the index of an initial basic variable. Since the right hand side of Equations 2.4-8 and 2.4-9 are equivalent, it is possible to save some computational effort.

2.5. Postoptimality Analysis by SFG

In most practical LP problems, some of the problem data are estimated and are not known exactly. The decision-makers are interested in knowing: 1) the range of problem data such that the optimal solution does not change, 2) what effect does an addition of a new decision variable or constraint have on the optimal solution. In particular the following variations in the problem will be considered.

a. Changes in the profit coefficients (C_j).

b. Changes in the resource constants (b_i).

c. Changes in the technological coefficients (a_{ij}).

d. Adding a new decision variable (X_j).

e. Adding a new constraint.

In this section, we shall see how to minimize the additional computations necessary to study the above changes by SFG procedure. First, consider an example problem.

<u>Example 2.5.1:</u>

$$\max Z = 5X_1 + 4X_2 + 3X_3$$

$$\text{s.t.} \quad X_1 + 3X_2 + 4X_3 + X_4 = 6$$

$$2X_1 - X_2 + X_3 + X_5 = 4$$

$$X_j \geq 0 \quad j = 1,5$$

The final SFG of this problem is shown below.

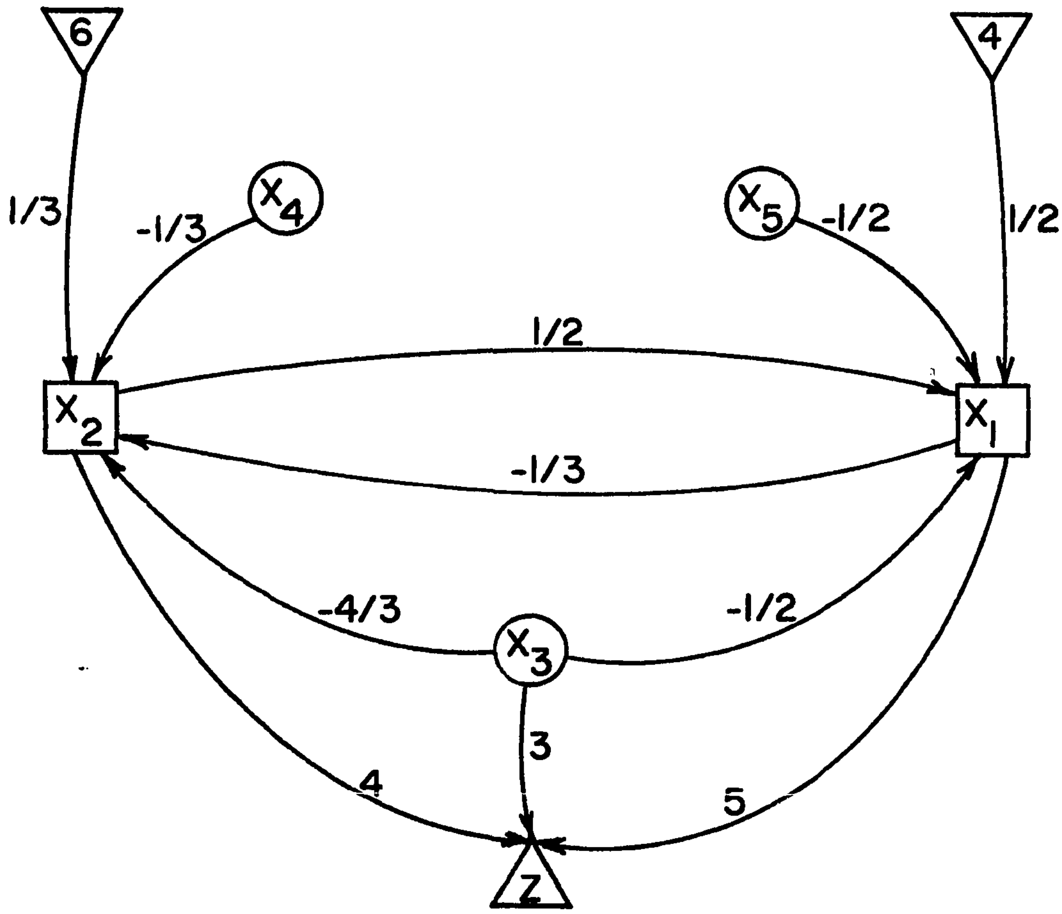


Figure 2-7. Final SFG of Example 2.5.1

Where the optimal solution is:

$$X_1 = \frac{18}{7}; \quad X_2 = \frac{8}{7}; \quad X_3 = X_5 = 0; \text{ and } Z = \frac{122}{7} .$$

The above example will be used to illustrate the SFG procedure of postoptimality analysis.

2.5.1. Changes in the profit coefficients (C_j)

Variations in the profit coefficients of the objective function may occur either in the profit of basic or nonbasic variables. These two cases will be treated separately.

Case 1: Changes in the profit coefficient of a basic variable.

Suppose the decision-maker is interested in knowing the effect of changes on the profit coefficient of basic variable X_1 of Example 2.5.1. It is clear that the variation on C_1 from its present value of 5 might change the composition of the optimal solution. To determine the range of variation on C_1 such that the optimal basis does not change, the transmittance of the branch from X_1 to Z, on the final SFG is replaced by C_1 and the transmittance from all nonbasic variables to the objective variable Z is calculated. In order for the optimal basis to remain unchanged, all these transmittances must be less than or equal to zero. In other words

$$T_{X_1 \rightarrow Z} \leq 0, \quad k \varepsilon K \qquad (2.5\text{-}1)$$

Referring to Figure 2-7 the range of variation on C_1 can be determined using Equation 2.5-1.

$$T_{X_3 \to Z} = \frac{3(1+\frac{1}{6})+(-\frac{1}{2})(C_1)+(-\frac{1}{2})(-\frac{1}{3})(4)+(-\frac{4}{3})(4)+(-\frac{4}{3})(\frac{1}{2})(C_1)}{1+\frac{1}{6}}$$
$$\leq 0$$

$$\Longrightarrow \quad C_1 \geq -1$$

$$T_{X_4 \to Z} = \frac{(-\frac{1}{3})(4)+(-\frac{1}{3})(\frac{1}{2})(C_1)}{1+\frac{1}{6}} \leq 0 \Rightarrow C_1 \geq -8$$

$$T_{X_5 \to Z} = \frac{(-\frac{1}{2})(C_1)+(-\frac{1}{2})(-\frac{1}{3})(4)}{1+\frac{1}{6}} \leq 0 \Rightarrow C_1 \geq \frac{4}{3}$$

Since each inequality must be satisfied to maintain optimality, $C_1 \geq \frac{4}{3}$.

Hence, Figure 2-7 remains unchanged as long as $C_1 \geq \frac{4}{3}$.

Case 2: Changes in the profit coefficient of a nonbasic variable.

One might be interested in the range of variations on a nonbasic variable such that the optimal solution remains optimal. As an example consider the range of C_3 of Example 2.5-1. We use the final SFG of Figure 2-7, and replace the transmittance of the branch between X_3 and Z. In order for the optimal basis to remain the same $T_{X_3 \to Z}$ must be less than or equal to zero. From Figure 2-7

$$T_{X_3 \to Z} = \frac{C_3(1+\frac{1}{6})+(-\frac{1}{2})(5)+(-\frac{1}{2})(-\frac{1}{3})(4)+(-\frac{4}{3})(4)+(-\frac{4}{3})(\frac{1}{2})(5)}{1+\frac{1}{6}}$$

$$\leq 0 \Longrightarrow C_3 \leq 9.$$

Therefore, the optimal basis remains the same as long as $C_3 \leq 9$.

2.5.2. Changes in the resource constants (b_i)

In the Example 2.5.1 suppose we wish to determine the range of variations of one of the resources such that the final SFG of Figure 2-7 remains unchanged. Assume we are interested in the range of the first resource, b_1. The changes on b_1 should be limited to values for which we maintain the feasibility of the optimal basis, i.e. $X_B = B^{-1}b \geq 0$ or in SFG terminology using Equation 2.4-4

$$X_j = \sum_{i=1}^{m} (T_{b_i \to X_j}) b_i \geq 0 \quad \forall \ (j \,|\, X_j \text{ is basic})$$

Referring to Figure 2-7 by substituting b_1 for 6 in the left hand side resource node, and using the above condition we obtain

$$X_1 = (T_{b_1 \to X_1})(b) + (T_{4 \to X_1})(4) = \frac{b_1+12}{7} \geq 0 \Rightarrow b_1 \geq -12$$

$$X_2 = (T_{b_1 \to X_2})(b_1) + (T_{4 \to X_2})(4) = \frac{2b_1-4}{7} \geq 0 \Rightarrow b_1 \geq 2$$

Since each inequality must be satisfied to assure feasibility, $b_1 \geq 2$.

Thus Figure 2-7 remains unchanged as long as $b_1 \geq 2$.

2.5.3. Changes in the technological coefficients (a_{ij})

Consider Example 2.5.1, the equations of the final SFG of Figure 2-7 are:

$$Z = 5X_1 + 4X_2 + 3X_3 \tag{2.5-2}$$

$$X_2 = \frac{1}{3}(-X_1 - 4X_3 - X_4 + 6) \tag{2.5-3}$$

$$X_1 = \frac{1}{2}(X_2 - X_3 - X_5 + 4) \tag{2.5-4}$$

Two possible cases may be distinguished in the studying of the variations of a_{ij}: 1) a_{ij} is an entry of a nonbasic variable column, 2) a_{ij} is an element of a basic variable column. These two cases will be treated separately.

<u>Case 1</u>: Changes in the technological coefficient of a nonbasic variable.

Suppose the decision-maker needs to know the range of coefficient of X_3 in the Equation 2.5-3 (i.e., a_{13}), for which the optimal basis remains the same. It is clear that the variation of a_{13} will have an effect on $\overline{C} = C_3 - C_B B^{-1} P_3$, and in order for the basis to remain unchanged $\overline{C}_3$ must be nonpositive. Using Equation 2.4-6, and replacing the transmittance of the branch between nodes X_3 and X_2 of Figure 2-7 by $-a_{13}/3$ we obtain:

$$T_{X_3 \to Z} = \frac{3(1+\frac{1}{6})+(-\frac{1}{2})(5)+(-\frac{1}{2})(-\frac{1}{3})(4)+(-\frac{a_{13}}{3})(4)+(-\frac{a_{13}}{3})(\frac{1}{2})(5)}{1+\frac{1}{6}}$$

$$\leq 0 \Longrightarrow a_{13} \geq \frac{10}{13}$$

Hence, the optimal basis remains the same for value $a_{13} \geq \frac{10}{13}$.

Case 2: Changes in the technological coefficient of a basic variable.

When the a_{ij} entry of a basic variable column changes, the range of a_{ij} is of no interest to us. But rather we are interested in knowing whether the optimal basis will remain unchanged. For example, we might want to know the effect of a new coefficient of X_1 in the second constraint (a_{21}) of Example 2.5.1, on the optimal basis. A new value for a_{21} alters the composition of B^{-1}, and perhaps it is easier to solve the problem from the beginning rather than using the final SFG.

2.5.4. Adding a new decision variable (X_j)

Suppose that a new decision variable X_{n+1} with unit profit C_{n+1} and technological coefficient column a_{n+1} is considered to be added to the present decision variables of Example 2.5.1. The final SFG Equations 2.5-2, 2.5-3 and 2.5-4 must be modified to accommodate this addition.

Modifying Equations 2.5-2 and 2.5-3 and 2.5-4 we obtain:

$$Z = 5X_1 + 4X_2 + 3X_3 + C_6X_6 \tag{2.5-5}$$

$$X_2 = \frac{1}{3}(-X_1 - 4X_3 - X_4 - a_{16}X_6 + 6) \tag{2.5-6}$$

$$X_1 = \frac{1}{2}(X_2 - X_3 - X_5 - a_{26}X_6 + 4) \tag{2.5-7}$$

Using the above equations, the final SFG of Figure 2-7 should be changed to represent the addition. This will be accomplished by adding a new nonbasic node for X_6, and three branches emanating from it. These branches will originate at X_6 and terminate at Z, X_2, and X_1 with transmittances C_6, $-a_{16}$, and $-a_{26}$ respectively. Now we have to determine whether X_6 is a candidate to enter the basis, and this can be done by evaluating $T_{X_6 \to Z}$. If $T_{X_6 \to Z}$ is non-positive, then the optimal basis will not change, on the other hand if $T_{X_6 \to Z}$ is positive, we proceed with usual SFG iterations until an optimal solution is attained (if one exists).

2.5.5. Adding a new constraint

Addition of a new constraint can affect the feasibility of the optimal solution only if it "cuts away" the optimal point, that is, the new constraint is not satisfied. Thus, the first step is to check whether the new constraint is satisfied by the present optimal solution. If it is satisfied then the optimal basis does not change, and the constraint is redundant. Otherwise, the final SFG should be modified to accommodate the new constraints and then feasibility of the new solution should be checked. For example, suppose in Example 2.5.1 a new constraint $a_{31}X_1 + a_{32}X_2 + a_{33}X_3 + X_6 = b_3$ is added to the set of constraints. Further, suppose X_6 is a slack variable with $C_6 = 0$. Treating X_6 as the dependent variable we will add a new basic variable node to the SFG of Figure 2-7 with the incoming branches to this node originating from nodes X_1, X_2, and X_3. Using Equation 2.4-4 we can check the feasibility of X_6, which should be infeasible. Now we can use dual simple method to achieve feasibility. In this case X_6 is a candidate to leave the basis, and to determine the incoming variable we use the following equation

$$\max\left(\frac{T_{X_j \to Z}}{T_{X_i \to X_j}} ; T_{X_i \to X_j} < 0\right) \tag{2.5-8}$$

where X_i is the leaving basic variable, and $(j|X_j$ is non-basic).

The validity of Equation 2.5-8 can be verified by using a similar procedure as described in Section 2.3.

3. THE MELLIN TRANSFORM

Mellin transforms are useful in transforming the products and quotients of functions into algebraic form, and been used in solving the nonlinear differential equations.

In this chapter we attempt to explain the properties of the Mellin transform. In particular, Section 3.1 states the fundamental characteristics of the Mellin transform, and Section 3.2 demonstrates the properties of the Mellin transform in the field of statistics. Properties 1 through 7 are based on the work of the earlier authors (e.g., Epstein (1948)), and properties 8 through 12 are extensions of the previous results.

3.1. Fundamental Characteristics

The Mellin transform of a continuous positive function f(x) is defined as:

$$M(f(x)) \triangleq F_f(x|S) \triangleq \int_0^\infty x^{s-1}\, f(x)\,dx \qquad x \geq 0 \qquad (3.1\text{-}1)$$

where s is a complex variable.

The inversion formula to recover f(x) given $F_f(x|s)$ is

$$f(x) = \frac{1}{2\pi i} \lim_{b\to\infty} \int_{a-ib}^{a+ib} X^{-s} F_f(x|s)\,ds \qquad (3.1\text{-}2)$$

where $i = \sqrt{-1}$, and integration is over the complex plane. In practice, the analytic recovery of $f(x)$ from $F_f(x|s)$ may become very complicated if not impossible. However, numerical integration often can be used to recover $f(x)$. The Mellin convolution of two continuous functions $f_1(x)$ and $f_2(x)$, $0 \leq x < \infty$, is defined as

$$h(x) = \int_0^{\infty} \frac{1}{y} f_2\left(\frac{x}{y}\right) f_1(y)\,dy \qquad (3.1\text{-}3)$$

When applying statistics to real world problems there is often a need to determine the probability distribution of the product of two positive independently distributed random variables. The following proposition provides the desired probability distribution.

Proposition 3.1.1:

Suppose the positive independent random variables X and Y are distributed in accordance with the continuous probability distributions $f(x)$ and $g(y)$, respectively. Then the distribution of random variable $R = X.Y$ is given by:

$$h(r) = \int_0^{\infty} \frac{1}{y} f(x/y) g(y)\,dy \qquad (3.1\text{-}4)$$

Proof:

Since x and y are independent, the joint probability distribution of x and y is $f_{xy}(x,y) = f(x).g(y)$. Let the dummy random variable T be equal to random variable y that is

$T = y$ implies $X = \frac{R}{T}$. The joint probability distribution of T and R is

$$h(t,r) = f(\tfrac{r}{t}).g(t).|J|$$

Where the Jacobian J is the 2x2 determinant

$$J = \begin{pmatrix} \frac{\partial X}{\partial R} & \frac{\partial X}{T} \\ \\ \frac{\partial Y}{\partial R} & \frac{\partial Y}{\partial T} \end{pmatrix}$$

$$J = \begin{pmatrix} \frac{1}{T} & \frac{-R}{T^2} \\ \\ 0 & 1 \end{pmatrix} = \frac{1}{T} .$$

Therefore,

$$f(t,r) = \frac{1}{T}f(\tfrac{r}{t}).f(t)$$

Since we are interested in the marginal distribution of R, we integrate the joint distribution with respect to t.

$$h(r) = \int_0^\infty \frac{1}{t}f(\tfrac{r}{t}).g(t)dt$$

Replacing $r = x.y$ and $t = y$ into the above equation we obtain Equation 3.1-4.

The right hand side of Equation 3.1-4 can be viewed as the Mellin convolution of the functions f(x) and g(y). Thus, the Mellin transform can be useful in studying the product of random variables.

3.2. Mellin Transform in Statistics

Fourier and Laplace transforms have extensively been used in statistics as a powerful analytical tool in studying the distribution of sums of independent random variables. As Epstein (1948) points out the Mellin transform is the counterpart of Fourier and Laplace transforms in studying the distribution of products and quotients of independent random variables. Epstein (1948) also states that like the Fourier transform, the Mellin transform has the desirable property that there is a one to one correspondence between a probability density function and its transform. Epstein (1948), Fox (1957), Springer and Thompson (1966, 1970) have shown some of the properties of Mellin transform in statistics. These properties are stated below.

Properties of Mellin transform in statistics:

1. The Mellin transform of a positive random variable X with continuous p.d.f. f(x) is:

$$F_f(x|s) \equiv E[X^{s-1}] = \int_0^\infty x^{s-1} f(x)dx \qquad Re(s) > 0$$

Where E is the expected value operator.

2. Scalar factor:

Let $Y = ax$

Where $a>0$ is a constant and random variable X is as in 1, the Mellin transform of y is:

$$F_f(y|s) \equiv E[y^{s-1}] = E[a^{s-1}x^{s-1}] = a^{s-1} F_f(x|s) \quad Re(s) > 0.$$

3. Products of n independent random variables.

Let $y = \prod_{i=1}^{n} X_i$ where X_i each have p.d.f. $f_i(x)$ with known Mellin transforms $F_{f_i}(X|s)$. The Mellin transform of y is:

$$F(y|s) \equiv E[y^{s-1}] = E[(\prod_{i=1}^{n} X_i)^{s-1}]$$

$$= E[X_1^{s-1}].E(X_2^{s-1}]\ldots E[X_n^{s-1}]$$

$$= \prod_{i=1}^{n} F_{f_i}(x|s) \quad Re(s) > 0.$$

4. Exponent:

Let $Y = X^a$, where a is a constant and X is as in 1.

The Mellin transform of Y is:

$$F_f(y|s) = E[Y^{s-1}] = E[X^{as-a}] = E[X^{(as-a+1)-1}]$$

$$= F_f(X|as-a+1) \quad Re(s) > 0.$$

In particular if $a = -1$, i.e., $y = 1/X$, then

$$F_f(Y|s) = E[Y^{s-1}] = F_f(x|-s+2).$$

5. Quotient of two independent random variables.

Let $y = X_1/X_2 = (X_1)(\frac{1}{X_2})$, where X_1 and X_2 have p.d.f.s $f_1(x)$ and $f_2(x)$, with known Mellin transforms $F_{f_1}(x|s)$ and $F_{f_2}(x|s)$, respectively. The Mellin transform of y is:

$$F(y|s) = E[y^{s-1}] = E[((X_1)(\frac{1}{X_2}))^{s-1}]$$

$$= E[X_1{}^{s-1}]\cdot E[(X_2{}^{-1})^{s-1}] = F_{f_1}(x|s).F_{f_2}(x|2-s)$$

$$\mathrm{Re}(s) > 0.$$

6. Area under p.d.f.

$$\text{Area} = \int_0^\infty f(x)dx = \int_0^\infty x^{1-1}f(x)dx = F_f(X|s)\Big|_{s=1} = 1$$

7. Moments:

$$E[X] = \int_0^\infty x\ f(x)dx = F_f(x|s)\Big|_{s=2} = F_f(x|2)$$

$$\mathrm{Var}(X) = F_f(x|3) - [F_f(x|2)]^2$$

In general $F_f(x|s)$ can be considered as the (s-1)th moment of X. Thus the kth moment about zero is given by $E[X^k] = F_f(x|s)\Big|_{k+1}$.

8. Cumulative distribution function (CDF)

Let $C(X) = \int_0^X f(t)dt$

where f(t) is the p.d.f. of random variable X. Denote the Mellin transform of C(x) by $F_c(x|s)$. Then by definition we can write

$$F_c(x|\alpha) = \int_0^\infty x^{\alpha-1} \int_0^x f(t)dt\, dx.$$

Integrate by parts. Let

$$u = \int_0^x f(t)dt \quad \text{and } dv = x^{\alpha-1}dx.$$

Then

$$du = f(x)dx \text{ and } v = \frac{1}{\alpha} x^\alpha.$$

Thus,

$$F_c(X|\alpha) = [\frac{1}{\alpha} x^\alpha \int_0^x f(t)dt] \Big|_0^\infty - \int_0^\infty \frac{1}{\alpha} x^\alpha f(x)dx.$$

Since the

$$\lim_{x\to 0} [\frac{1}{\alpha} x^\alpha \int_0^x f(t)dt] = 0 \quad \text{and} \quad \text{for } -1<Re(\alpha)<0$$

$$\lim_{x\to\infty} [\frac{1}{\alpha} x^\alpha \int_0^x f(t)dt] = 0$$

Note that for $-1<Re(\alpha)<0$ the function $F_c(x|s)$ is analytic. Thus the function

$$F_c(X|\alpha) = -\int_0^\infty \frac{1}{\alpha} x^\alpha f(x)dx. \qquad -1<\mathrm{Re}(s)<0$$

Let

$$\alpha = \beta-1$$

then

$$F_c(X|\beta-1) = -\frac{1}{\beta-1}\int_0^\infty x^{\beta-1} f(x)dx$$

$$= -\frac{1}{\beta-1} F_f(X|\beta). \qquad 0<\mathrm{Re}(\beta)<1$$

Note that when

$$\beta-1 = s$$

then

$$F_c(X|s) = -\frac{F_f(X|s+1)}{s} \qquad -1<\mathrm{Re}(s)<0$$

Similarly it can be shown that

$$F_{(1-c)}(X|s) = \frac{F_f(X|s+1)}{s} \qquad -1<\mathrm{Re}(s)<0$$

where

$$(1-c(x)) = \int_x^\infty f(t)dt.$$

9. Truncated cumulative distribution function

Let the cumulative distribution function of p.d.f. $F(x)$ be $c(x) = \int_0^X f(t)dt$. Denote the truncated $C(X)$ by $W(X;\ U,\ L)$ where L and U are the specified lower and upper bounds respectively, $0\leq L<U$. That is:

$W(X;\ u,L) = \int_L^U f(X)dx$ which can be put in the following form:

$W(X;\ U,\ L) = C(X)\ H(x-L) - C(x)H(x-U)$ where $H(X)$ is a unit step function. The Mellin transform of product of two functions $f(x)$ and $g(x)$ is given by Carrier et al. (1966) as:

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F_f(X|s-\tau)\ F_g(X|\tau)d\tau.$$

Recalling that the Mellin transform of a unit step function $H(x-\alpha)$ is $-\frac{\alpha^s}{s}$, and invoking the Carrier et al. (1966) result we obtain the Mellin transform of $W(x;\ U,\ L)$.

$$F_W(x;\ U,L|S) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{U^{s-\tau}-L^{s-\tau}}{s-\tau}\right)F_c(X|\tau)d\tau.$$

10. Determination of semi-variance from Mellin transform.

Markowitz (1959) introduced the notion of semi-variance (SV) as an alternative measure of degree of variability and skewness of a distribution. In general semi-variance is defined as:

$$SV \triangleq \int_0^{\mu} (x-\mu)^2\ f(x)dx \qquad (3.2\text{-}1)$$

where μ is the mean of probability density function $f(x)$. Markowitz (1959) proposed that one might use the ratio

$\frac{\text{Var.}}{2\text{SV}}$ as a measure of skewness. For symmetric distributions the ratio is one; if a distribution is skewed to the right, then the ratio is greater than one; and if the skewness is to the left, the ratio will be less than one. (Markowitz (1959) gives the advantages and disadvantages of using semi-variance vs. variance in a portfolio selection situation.)

The SV can be written mathematically in the following form. By definition

$$SV = \int_0^{\mu} (x-\mu)^2 f(x)\,dx$$

If we let $H(x)$ and $H(x-\mu)$ represent two unit step functions the SV can be represented as:

$$SV = \int_0^{\infty} (x-\mu)^2 f(x)\,[H(x)-H(x-\mu)]\,dx$$

which reduces to

$$SV = \text{Var}(x) - \int_0^{\infty} (x-\mu)^2 H(x-\mu)\,f(x)\,dx. \qquad (3.2\text{-}2)$$

Assume the Mellin transform of $f(x)$ is known to be $F_f(x|s)$; then the $\text{Var}(x)$ is: $\text{Var}(x) = F_f(x|3) - (F_f(x|2))^2$. The representation of the integral in Equation 3.2-2 in terms of $F_f(x|s)$ is a bit more complicated. Carrier et al. (1966) give the following property of the Mellin transform:

$$\int_0^\infty k(x)g(x)dx = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F_k(x|s)F_g(x|1-s)ds. \qquad (3.2\text{-}3)$$

In view of Equation 3.2-3 let

$$k(x) \equiv (x-\mu)^2 H(x-\mu).$$

Since the Mellin transform of $x^n\ H(x-\mu)$ is equal to $-\frac{\mu^{s+n}}{s+n}$, the Mellin transform of $k(x)$ can be written as:

$$F_k(x|s) = -\frac{\mu^{s+2}}{s+2} + \frac{2^{s-1}\mu^{2(s-1)}}{s+1} - \frac{\mu^{3s-2}}{s}. \qquad (3.2\text{-}4)$$

Using Equation 3.2-3 and 3.2-4 the Equation 3.2-2 can be written as:

$$SV = F_f(x|3) = (F_f(x|2))^2$$
$$-\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(-\frac{\mu^{s+2}}{s+2} + \frac{2^{s-1}\mu^{2(s-1)}}{s+1} - \frac{\mu^{3s-2}}{s}\right)F_f(x|1-s)ds.$$

(3.2-5)

Note that Equation 3.2-5 gives the SV in terms of the Mellin transform of p.d.f. $f(x)$.

11. Calculating mean and variance of summation of independent random variables from their Mellin transforms.

Let $Z = X+Y$ where X and Y are independent random variables with probability density functions $f(x)$ and $f(y)$, respectively.

It can be shown that random variable Z has the p.d.f. $f(z) = \int_0^\infty f_x(z-y)f_y(y)dy$. Therefore, the Mellin transform of f(z) can be written as:

$$F_f(z|s) = \int_0^\infty z^{s-1}f(z)ds = \int_0^\infty \int_0^\infty z^{s-1}f_x(z-y)f_y(y)dydz.$$

Evaluation of the right hand side of the above equation becomes quite involved. Since we are interested in the lower moments of random variable Z, rather than taking a direct approach, let us find the moments of random variable Z. This can be accomplished by evaluating the terms of the moments of its components as calculated from the components' Mellin transforms. From standard probability theory we can show that E[z] = E[x] + E[y].

$$E[z] = \int_0^\infty \int_0^\infty (x+y)f(x).f(y)dxdy$$

$$= \int_0^\infty xf(x)dx \int_0^\infty f(y)dy + \int_0^\infty yf(y)dy \int_0^\infty f(x)dx$$

$$= E[x] + E[y] \tag{3.2-6a}$$

If x and y are positive random variables, from property 7 we know that

$$E[x] = F_f(x|2) \text{ and } E[y] = F_f(y|2)$$

Replacing the above in Equation 3.2-6a we obtain

$$E[z] = F_f[z|2] = F_f(x|2) + F_f(y|2) \qquad (3.2\text{-}6b)$$

Also using standard probability theory we can show that

$E[z^2] = E[x^2] + E[y^2] + 2E[x]E[y]$.

$$E[z^2] = \int_0^\infty \int_0^\infty (x+y)^2 \; f(x)f(y)dxdy$$

$$= \int_0^\infty \int_0^\infty (x^2+y^2+2xy)f(x)f(y)dxdy$$

$$= \int_0^\infty x^2 f(x)dx \int_0^\infty f(y)dy + \int_0^\infty y^2 f(y)dy \int_0^\infty f(x)dx$$

$$+ 2\int_0^\infty \int_0^\infty x.y \; f(x)f(y)dxdy$$

$$= E[x^2]+E[y^2]+2E[x].E[y]. \qquad (3.2\text{-}7)$$

Because x and y are independent.

If x and y are positive random variables, from property 7 we can write

$$E[x^2] = F_f(x|3); \quad E[y^2] = F_f(y|3)$$

$$E[x] = F_f(x|2); \; E[y] = F_f(y|2)$$

Substituting the above in Equation 3.2-7 we get,

$$E[z^2] = F_f(x|3) + F_f(y|3) + 2F_f(x|2).F_f(y|2).$$

In general, if $Z = \sum_{i=1}^{n} X_i$, $X_i \geq 0$ such that X_is are independently distributed with probability density functions $f_i(x)$ and Mellin transforms $F_{f_i}(x|s)$ then, we can state the following relationships:

$$E[z] = \sum_{i=1}^{n} F_{f_i}(x|2) \qquad (3.2\text{-}8)$$

and

$$E[z^2] = \sum_{i=1}^{n} F_{f_i}(x|3) + 2 \sum_{j=1}^{n-1} \sum_{i=j}^{n} F_{f_j}(x|2) F_{f_i}(x|2) \quad j=1,n-2. \qquad (3.2\text{-}9)$$

The variance of z can be determined using Equations 3.2-8 and 3.2-9 since

$$\mathrm{Var}(z) = E[z^2] - (E[z])^2$$

using the same procedure the higher moments of z can also be found in terms of the moments of each component as calculated from the Mellin transform for each of these components.

12. Calculating mean and variance of the summation of random variables which are perfectly correlated, from their Mellin transforms.

Let $Z = X+Y$ where X and Y are not independent random variables and define the probability density functions as $f(x)$ and $f(y)$, for X and Y, respectively.

Then

$$E[Z] = E[X] + E[Y]$$

$$\begin{aligned} E[Z^2] &= E[(X+Y)^2] \\ &= E[X^2+Y^2+2XY] \\ &= E[X^2] + E[Y^2] + 2E[XY] \end{aligned}$$

and

$$\begin{aligned} Var(Z) = E[X^2] + E[Y^2] + 2E[XY] - E[X]^2 - E[Y]^2 \\ - 2E[X].E[Y] \end{aligned}$$

or equivalently

$$Var(Z) = Var(X) + Var(Y) + 2Cov(XY). \quad (3.2\text{-}10)$$

Now assume that X and Y are perfectly (linearly) correlated, that is, the correlation coefficient $\rho=1$, then we can represent Y as a linear function of X, i.e. $Y = a+bX$ where a and b are constants. Thus the covariance of X and Y becomes:

$$\begin{aligned} Cov(XY) &= E[XY] - E[X].E[Y] \\ &= E[X(a+bX)] - E[X].E[a+bX] \\ &= aE[X]+bE[X^2]-aE[X]-bE[X]^2 \\ &= b(E[X^2]-E[X]^2) \\ &= b\ Var(X) \qquad (3.2\text{-}11) \end{aligned}$$

Substituting Equation 3.2-11 into Equation 3.2-10, the variance of Z in terms of variance of X is

$$Var(Z) = (1+b)^2\ Var(x) \qquad (3.2\text{-}12)$$

This can be generalized to a sum of n.m variables: Let X_i (i=1,n) represent an independent random variable with p.d.f. $f_i(x)$. Suppose the random variable Y_{ij} (j=1,m) denotes the jth variable which is linearly correlated with the ith (i=1,n) independent random variable. Such that: $Y_{ij} = a_j + b_j X_i$. Then

$$E[y_{ij}] = a_j + b_j . E[X_i]$$

$$Var[Y_{ij}] = b_j^2 Var[X_i]$$

Define

$$Z_i = X_i + \sum_{j=1}^{m} Y_{ij} \qquad i=1,n$$

and

$$Z = \sum_{i=1}^{n} Z_i$$

Since X_i's are independent, the Z_i's will also be independent random variables. Then

$$E[Z_i] = E[X_i] + \sum_{j=1}^{m} E[y_{ij}]$$

$$= E[X_i] + \sum_{j=1}^{m} (a_j + b_j E[X_i])$$

$$= \sum_{j=1}^{m} a_j + (1 + \sum_{j=1}^{m} b_j) E[X_i]$$

$$Var[Z_i] = Var(X_i) + \sum_{j=1}^{m} Var(Y_{ij}) + 2Cov(X_i \ . \sum_{j=1}^{m} Y_{ij}).$$

But the $\mathrm{Cov}(X_i \cdot \sum_{j=1}^{m} Y_{ij}) = \sum_{j=1}^{m} b_j \, \mathrm{Var}(X_i)$. Therefore,

$$\mathrm{Var}(Z_i) = (1 + \sum_{j=1}^{m} b_j)^2 \, \mathrm{Var}(X_i).$$

This yields the following results for the mean and variance:

$$E[Z] = \sum_{i=1}^{n} E[Z_i]$$

$$= n \sum_{j=1}^{m} a_j + (1 + \sum_{j=1}^{m} b_j) \sum_{i=1}^{n} E[X_i] \qquad (3.2\text{-}13a)$$

and

$$\mathrm{Var}(Z) = \sum_{i=1}^{n} \mathrm{Var}(Z_i) = (1 + \sum_{j=1}^{m} b_j)^2 \sum_{i=1}^{n} \mathrm{Var}(X_i). \qquad (3.2.\text{-}14a)$$

Hence, if all X_i's and Y_i's (i=1,n; j=1,m) are positive random variables, using property 7 of Mellin transforms we can calculate the mean and variance of Z.

$$E[Z] = \sum_{i=1}^{n} [\sum_{j=1}^{m} F(a_j|2) + (1 + \sum_{j=1}^{m} F(b_j|2))F(X_i|2)] \qquad (3.2\text{-}13b)$$

$$\mathrm{Var}(Z) = (1 + \sum_{j=1}^{m} F(b_j|2))^2 \sum_{i=1}^{n} (F(X_i|3)-(F(X_i|2))^2 \qquad (3.2\text{-}14b)$$

Appendix B shows the Mellin transform of selected probability density functions.

4. STOCHASTIC LINEAR PROGRAMMING

The primary emphasis of this chapter is the development of new insights with regard to the complex nature of stochastic linear programming. In particular, we will address the models with stochastic profit vector (C), and stochastic resource vector (R). Also some decision rules for changing the basic vector are discussed.

4.1. Preliminaries

Consider the following linear programming problem stated as:

LP1

$$\max Z = CX$$

$$\text{s.t.} \quad AX = r \qquad (4.1\text{-}1)$$

$$X \geq 0 \qquad (4.1\text{-}2)$$

where:

X is an nx1 decision vector

C is an 1nx profit vector

r is an mx1 resource vector

A is a mxn technological coefficient matrix.

Following standard linear programming notation, let $X_B^{\ell} = (X_1^{\ell}, \ldots, X_m^{\ell})$ be the ℓth basis of LP1. Also, let C_B^{ℓ} and B^{ℓ} denote the ℓth profit vector and basis matrix of LP1, respectively. The ℓth basic solution is said to be feasible if $X_B^{\ell} = (B^{\ell})^{-1} r \geq 0$. Denoting the ijth entry of $(B^{\ell})^{-1}$ by λ_{ij}^{ℓ}, the ℓth basic feasible solution can be written as:

$$X_i^{\ell} = \sum_{j=1}^{m} \lambda_{ij}^{\ell} r_j \quad \forall\ i \varepsilon m \tag{4.1-3}$$

Substituting the above equation in Equation 4.1-1 we obtain:

$$Z^{\ell} = \sum_{i=1}^{m} C_i^{\ell} X_i^{\ell} = \sum_{i=1}^{m} \sum_{j=1}^{m} \lambda_{ij}^{\ell}\ C_i^{\ell} r_j^{\ell} \tag{4.1-4}$$

where Z^{ℓ} is the value of the objective function associated with the ℓth basis. If elements of A, C and r are fixed known constants, then by the widely used simplex algorithm, or some other solution methods, in a finite number of steps the optimal solution to LP1 can be determined. Now if we assume one or all elements of vectors C and/or R are random variables, then the deterministic LP solution methods will not provide us with sufficient information about the stochastic nature of the problem. The assumption of vectors C and/or R being stochastic tends to agree with the behavior of most of the real world problems, since the per-unit profit and availability of resources are a function of market economy. Conversely, the elements of matrix A are, in general, some measured data, and their variations are often due to errors of measurements. The effect of these errors on the optimal solution can be determined by the standard LP sensitivity analysis. From theory of LP we know that there exists $K = \frac{n!}{m!(n-m)!}$ possible bases. Assuming that the solution given by the ℓth basis ($\ell \epsilon K$) has been determined to be, in some sense, an optimal solution, then we can consider the following three cases:

<u>Case I: C is stochastic; r and A are fixed:</u>

Let the positive random variable C_i be the ith element of profit vector C. Assume the C_i's are independently distributed and their probability density functions are known to be $f_i(C)$, $(i=1,n)$. Since $(B^{-1})^{\ell}$ and r are fixed by assumption, the X_B^{ℓ} is known with probability of one. But the value of the objective function $Z^{\ell} = \sum_{i=1}^{m} C_i^{\ell} X_i^{\ell}$ will be stochastic in nature. At least in theory the distribution of Z^{ℓ} can be determined. If we are interested in moment generating function (M.G.F) of Z^{ℓ}, $(M_{Z^{\ell}}(\alpha))$ and its respective moments, then the convolution property of M.G.F. for independent random variables can be utilized to obtain the expression of $M_{Z^{\ell}}(\alpha)$.

$$M_{Z^{\ell}}(\alpha) = E[e^{(\sum_{i=1}^{m} X_i C_i)\alpha}] = \prod_{i=1}^{m} M_{C_i}(X_i^{\ell}\alpha) \qquad (4.1\text{-}5)$$

where

$M_{C_i}(\alpha)$ is the M.G.F. of C_i.

By subsequent differentiations of Equation 4.1-5 and evaluation at $\alpha=0$, the moments of Z^{ℓ} can be determined. If we are only interested in the lower moments of Z^{ℓ}, an alternative will be to determine an expression of Mellin transforms denoted by $Z^{\ell}(s)$. Suppose the Mellin transforms of $F_i(C)$ exist and are represented by $F_{f_i}(C|s)$. Then

$$Z^{\ell}(s) = \sum_{i=1}^{m} F_{f_i}(C|s) \cdot F(X_i^{\ell}|s) \qquad (4.1\text{-}6)$$

where $F(X_i^{\ell}|s)$, $(i=1,m)$ are the Mellin transform of the basic variables. By using property 11 of the Mellin transforms the first and second moments of Z^{ℓ} can be determined. One advantage of the Mellin transform in calculating the moments of a distribution, unlike moment generating function, is the fact that no differentiation is necessary to obtain the moments. This may prove to be advantageous when a computer is used to solve a problem of this nature.

<u>Case II: R is stochastic; C and A are fixed:</u>

Suppose the positive random variable R_i is the ith element of stochastic resource vector R. Let R_i's be independently distributed with known probability density function $g_i(r)$; $(i=1,m)$. In view of Equations 4.1-3 and 4.1-4, it is apparent that X_i^{ℓ} and Z^{ℓ} both are random variables, and dual variables $Y^{\ell} = C^{\ell}(B^{-1})^{\ell}$ are constants. Again by invoking the convolution property of moment generating function (M.G.F.), the moment generating function of X_i^{ℓ}, $(M_{X_i^{\ell}}(\alpha))$ and Z^{ℓ}, $(M_{Z^{\ell}}(\alpha))$ can be written as follows:

$$M_{X_i^\ell}(\alpha) = E[e^{(\sum_{j=1}^{m} \lambda_{ij}^\ell R_j)\alpha}] = \prod_{j=1}^{m} M_{R_j}(\lambda_{ij}^\ell \alpha) \qquad (4.1\text{-}7)$$

$$M_{Z^\ell}(\alpha) = E[e^{\sum_{i=1}^{m} \sum_{j=1}^{m} \lambda_{ij}^\ell C_i^\ell R_j^\ell}] = \prod_{j=1}^{m} M_{R_j}(\sum_{i=1}^{m} \lambda_{ij}^\ell C_i^\ell \alpha) \qquad (4.1\text{-}8)$$

where

$M_{R_j}(\alpha)$ is the M.G.F. of R_j, and

$Y_j^\ell = \sum_{i=1}^{m} \lambda_{ij}^\ell C_i^\ell$ is the shadow price of R_j,

from which the moments of X_i^ℓ and Z^ℓ can be calculated. Assuming the Mellin transforms of $g_i(r)$ are known, then transform expressions for X_i^ℓ and Z^ℓ can be written as:

$$X_i^\ell(s) = \sum_{j=1}^{m} F(\lambda_{ij}^\ell|s) \cdot F_{g_i}(R|s) \qquad (4.1\text{-}9)$$

$$Z^\ell(s) = \sum_{i=1}^{m} F(y_i^\ell|s) \cdot F_{g_i}(R|s) \qquad (4.1\text{-}10)$$

where:

$X_i^\ell(s)$ is the Mellin transform expression associated with X_i^ℓ

$Z^\ell(s)$ is the Mellin transform expression associated with Z^ℓ

$F_{g_i}(R|s)$ is the Mellin transform of $g_i(r)$

$F(Y_i^{\ell}|S)$ is the Mellin transform of Y_i^{ℓ}

The discussion about the advantage of the Mellin transform made in Case I also applies to this case.

<u>Case III: C and R are stochastic; A is fixed:</u> Let C_i and R_i be defined as in cases I and II respectively, with the assumption that C_i and R_i are independently distributed. The decision variables X_i^{ℓ} and Z^{ℓ} are both random variables, and the moments of X_i^{ℓ} can be calculated from either Equation 4.1-7 or 4.1-9. The random variable Z^{ℓ} is defined by:

$$Z^{\ell} = \sum_{i=1}^{m} \sum_{j=1}^{m} \lambda_{ij}^{\ell} C_i^{\ell} R_j^{\ell} \tag{4.1-11}$$

where C_i^{ℓ} and R_j^{ℓ} are both random variables. Unlike Cases I and II the moment generating function of Z^{ℓ} can not be found, unless we know the distribution of Z^{ℓ}. On the other hand for an important special case, using properties of the Mellin transform the lower moments of Z^{ℓ} can be found. Assume R_i and C_j ($i=1,m$; $j=1,n$) are independently distributed random variables such that $C_j = \alpha_{0j} + \alpha_{1j}\xi$ and $R_i = \beta_{0i} + \beta_{1i}\xi$ (where α_{0j}, α_{1j}, β_{0i}, and β_{1i} are constants), and further assume the p.d.f. of ξ is known to be $h(\xi)$, $\xi>0$. Under these assumptions the value of the objective function Z^{ℓ} is a function of the random variable ξ and is denoted by:

$$Z^{\ell} = K_0 + K_1\xi + K_2\xi^2 \tag{4.1-12}$$

where K_0, K_1, and K_2 are constant. By using properties of the Mellin transform (see Chapter 3) the mean and variance of Z^{ℓ} can be determined. So far in this section we have assumed that X^{ℓ} has been known to be, in some sense, an optimal solution. In the next section an attempt will be made to give some insight into the difficulty of selecting an optimal basis.

4.2. Optimality and Feasibility Conditions of SLP

In the classical linear programming problem, a solution is optimal (maximal) when $\overline{C}_j = C_j - C_B B^{-1} a_j \leq 0$; $(j \,|\, X_j$ is nonbasic) and it is feasible if $X_B = B^{-1} r \geq 0$. A feasible optimal solution is obtained when both of these conditions are satisfied. But due to the random nature of vectors C and R, at each optimal basis the linear program will have a probability of being feasible (P_f^{ℓ}), and one of being optimal (P_o^{ℓ}). Since by assumption C and R are independent random vectors the probability of the ℓth basis being a feasible optimal solution is $P_Z^{\ell} = P_f^{\ell} \cdot P_o^{\ell}$. Where the expressions for determining P_f^{ℓ} and P_o^{ℓ} are given below:

$$P_f^{\ell} = P\{\bigcap_{i=1}^{m} X_i^{\ell} \geq 0\} = P\{X_1 \geq 0\} \prod_{j=2}^{m} P\{X_j \geq 0 \,|\, \bigcap_{i=1}^{j-1} X_i \geq 0\} \tag{4.2-1}$$

$$P_o^{\ell} = P\{\bigcap_{j=m+1}^{n} \bar{C}_j \le 0\}$$

$$= P\{\bar{C}_{m+1} \le 0\} \prod_{j=m+2}^{n} P\{\bar{C}_j \le 0 \,\Big|\, \bigcap_{q=m+1}^{j-1} \bar{C}_{q_i} \le 0\} \qquad (4.2\text{-}2)$$

The distribution of the objective value under these conditions can be stated as:

$$Z(C,R) = \sum_{\ell=1}^{K} P_Z^{\ell} \cdot Z^{\ell}(C,R) \qquad (4.2\text{-}3)$$

where

$$K = \frac{n!}{m!(n-m)!}$$

Equation 4.2-3 implies that to determine distribution of the objective function, one must determine all K possible bases along with their respective probability of feasibility and optimality. However, in practice, the above proposition requires a tremendous amount of time and some difficult computational efforts. As an illustrative example for the amount of work involved, let us consider the following very simple example.

Example 4-1:

$$\text{Max } Z = 2X_1 + X_2$$

$$\text{s.t.} \quad X_1 + X_2 + X_3 = R_1$$

$$3X_1 + X_2 + X_4 = R_2$$

$$X_1, X_2, X_3, X_4 \geq 0$$

Where R_1 and R_2 are independent random variables with the uniform probability density functions: $R_1 \sim U(2,6)$, $R_2 \sim U(3,9)$. The K=6 possible bases are functions of R_1 and R_2 and are shown in Table 4-1. Figure 4-1 gives the graphical representation of Example 4-1. The probability that the ℓth ($\ell \leq K$) base is an optimal solution can be determined by using Equation 4.2-2, and since profit vector (C) is not a random vector, P_o^{ℓ} is zero or one. Since the bases are functions of R_1 and R_2, their respective probabilities of being feasible are also functions of R_1 and R_2. As a representative calculation let us find the probability that the first set of solution is feasible (P_f^1), and is optimal (P_o^1).

$$P_f^1 = P\{X_1 \geq 0\} \cdot P\{X_2 \geq 0 | X_1 \geq 0\}$$

$$P\{X_1 \geq 0\} = P\{(-R_1 + R_2) \geq 0\}$$

$$P\{X_2 \geq 0 | X_1 \geq 0\} = P\{(3R_1 - R_2) \geq 0 | (-R_1 + R_2) \geq 0\}$$

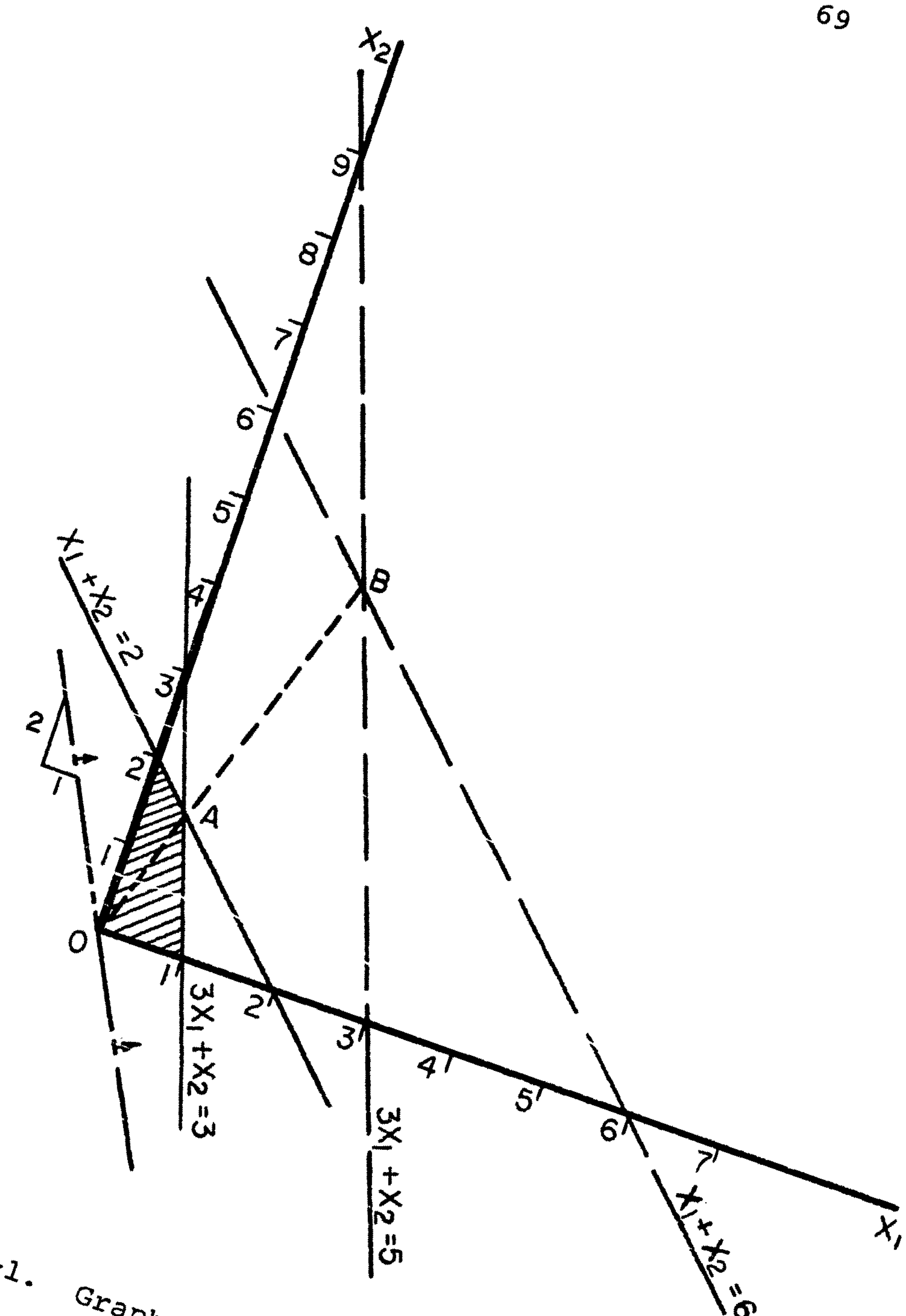


Figure 4-1. Graphical representation of Example 4-1

Table 4-1. Possible basis of Example 4-1

ℓ	X_1	X_2	X_3	X_4	Z^ℓ
1	$\frac{1}{2}(-R_1+R_2)$	$\frac{1}{2}(3R_1-R_2)$			$\frac{1}{2}R_1 + \frac{1}{2}R_2$
2	$\frac{1}{3}R_2$		$R_1 - \frac{1}{3} R_2$		$\frac{2}{3}R_2$
3	R_1			$-3R_1+R_2$	$2R_1$
4		R_2	R_1-R_2		R_2
5		R_1		$-R_1+R_2$	R_1
6			R_1	R_2	0

Referring to Figure 4-2:

$$P\{X_1 \geq 0\} = 1-\frac{1}{24}\int_3^6\int_{R_2}^6 dR_1 dR_2 = 1-\frac{1}{24}\int_3^6 (6-R_2)dR_2$$

$$= 1-\frac{1}{24}[6R_2-\frac{1}{2}R_2{}^2]\Big|_3^6 = \frac{39}{48}$$

$$P\{X_2 \geq 0 | X_1 \geq 0\} = \frac{39}{48} - \frac{1}{24}\int_6^9\int_2^{\frac{1}{3}R_2} dR_1 dR_2 = \frac{3}{4}$$

Therefore the probability of the first basis being feasible is:

$$P_f^1 = \frac{39}{48} \cdot \frac{3}{4} = \frac{39}{64} = 0.609375$$

In order to determine P_0^1, we have to calculate $\overline{C}_3$ and $\overline{C}_4$.

$$\overline{C}_3 = 0-(2 \quad 1)\begin{pmatrix} -1/2 & 1/2 \\ 3/2 & -1/2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = -1/2$$

$$\overline{C}_4 = 0-(2 \quad 1)\begin{pmatrix} -1/2 & 1/2 \\ 3/2 & -1/2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -1/2$$

$$P\{\overline{C}_3 \leq 0\} = 1$$

$$P\{\overline{C}_4 \leq 0 | \overline{C}_3 \leq 0\} = 1$$

Thus the probability that the first set is optimal is equal to one. Now the probability that Z^1 is feasible and optimal is

$$P_Z^1 = P_f^1 \cdot P_o^1 = 0.609375$$

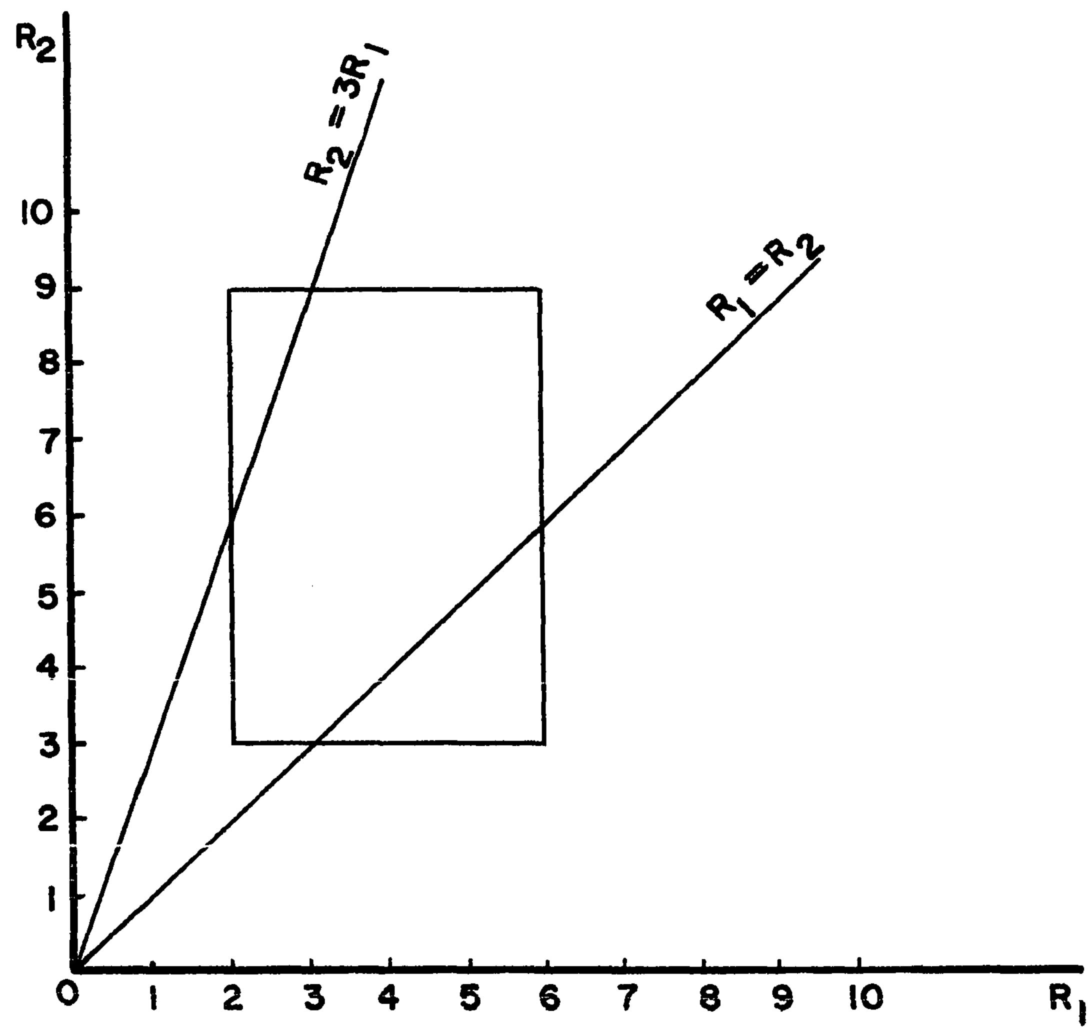


Figure 4-2. Feasible region of P_f^1

The probabilities associated with the other sets can be determined in the same manner. Table 4-2 shows the results for Example 4-1.

Thus in most practical situations, one might wish to sacrifice some exactness of the results for savings in computation time and ease of calculations. This implies that rather than checking all possible basis, we select one according to some predetermined decision rules. The idea of using simplex algorithm with a modified decision rule for changing the basis seems appealing. Some decision rules are as follows: a) use mean, b) use mode, c) use mean minus some constant multiplied by the standard deviation the constant could be a measure of risk aversion of the decision-maker, d) use mean minus semivariance (SV is defined in Chapter 3). These proposed rules are subject to further study and their applicabilities for particular situations need to be considered.

4.3. Balanced Stochastic Linear Program

Again, recall that the optimal feasible solution vector to a LP problem is $X_B^* = B^{-1}R$. X_B^* is a vector in the space R^m and as long as the slope of this vector is insensitive to the variations of the vectors R, C, and the matrix A individually, then the composition of the optimum basis will remain the same. In view of the above discussion, let us define a special class of stochastic linear programs.

Table 4-2. The results of Example 4-1 with the RHS of: $R_1 \sim U(2,6)$; $R_2 \sim U(3,9)$

ℓ	Basis	Probability that the basis is feasible	Probability that the basis is optimal	Value of the objective function	Prob. that the basis is feasible optimal
1	X_1&X_2	0.609375	1.0	$\frac{1}{2}(R_1+R_2)$	.609375
2	X_1&X_3	0.9375	0.0	$\frac{2}{3}R_2$	0.0
3	X_1&X_4	0.0625	1.0	$2R_1$	0.0625
4	X_2&X_3	0.1875	1.0	R_2	0.1875
5	X_2&X_4	0.8125	0.0	R_1	0.0
6	X_3&X_4	1.0	0.0	0	0.0

Definition 4.3.1: A stochastic linear program is called P-balanced when the elements of profit vector C are identically distributed to within linear transformations.

Definition 4.3.2: A stochastic linear program is called R-balanced when the elements of resource vector R are identically distributed to within linear transformations.

Definition 4.3.3: A stochastic linear program is called T-balanced when the entries of the technological matrix A are identically distributed to within linear transformations.

Definition 4.3.4: A stochastic linear program is called PRT-balanced if each of the definitions 4.3.1, 4.3.2, and 4.3.3 hold.

The following results reveal some of the properties of the balanced stochastic linear programs assuming an optimal solution exists. In particular, the following lemmas are proven for a special class of balanced SLP where the elements of vectors C, R, and matrix A are dependent random variables within their respective vector or matrix. Thus, we call this special class of balanced SLP "dependent balanced" stochastic linear program.

Lemma 4.3.1: For a given SLP, if the elements of random profit vector C are given by $C_i = C_{L_i}(1+C(\alpha))$; (i=1,n) and vector R and matrix A are constants where:

C_{L_i} is the lower bound of random variable C_i.

$$\alpha = \int_0^{C(\alpha)} f(C)dc; \quad 0\leq\alpha\leq 1,$$

Then the composition of the optimal basis X_B^* is insensitive to the random variation of vector C.

Proof:

Since r and A are constants, X_B^* will be unchanged as long as the slope of the hyperplane $Z^* = C_B X_B^*$ remains unchanged for the random variations of C_B. To show that the slope of Z^* does not change it is sufficient to show that the unit vector of vector C_B stays the same. The unit vector of C_B is defined as $U_{CB} = C_B/||C_B||$, where $||C_B||$ is the norm of C_B, or $U_{CB} = (\frac{C_1}{||C_B||}, \frac{C_2}{||C_B||}, \ldots, \frac{C_m}{||C_B||})$

for $\alpha=0$, that is C_B constant

$$U_{CB} = (\frac{C_{L_1}}{||C_B^0||}, \frac{C_{L_2}}{||C_B^0||}, \ldots, \frac{C_{L_m}}{||C_B^0||})$$

where

$$||C_B^0|| = (C_{L_1}^2 + C_{L_2}^2 + \ldots C_{L_m}^2)^{1/2}$$

For $0<\alpha\leq 1$

$$C_i = C_{L_i}(1+C(\alpha))$$

$$||C_B^{\alpha}|| = (1+C(\alpha))||C_B^0||$$

$$\vec{U}_{C_B}^{\alpha} = \frac{(1+C(\alpha))}{(1+C(\alpha))} \sum_{i=1}^{m} \frac{\vec{C}_{L_i}}{||C_B^0||} = \vec{U}_{C_B}^{o} \qquad \text{QED}$$

<u>Lemma 4.3.2</u>: For a given SLP, if the elements of random resource vector R are given by $R_i = R_{L_i}(1+R(\beta))$, and vector C and matrix A are constants where:

R_{L_i} is the lower bound of random variable R_i

$$\beta = \int_0^{R(\beta)} f(r)dr, \quad 0\le\beta\le1$$

Then the composition of the optimal basis X_B^* is insensitive to the random variations of R.

<u>Proof</u>: Since C and A are constants, the composition of X_B^* will remain unchanged as long as the slope of X_B^* does not change upon variation of random vector R. Again, we have to show that unit vector of X_B^* is insensitive to $R(\beta)$, and the rest of the proof is similar to Lemma 4.3.1.

<u>Lemma 4.3.3</u>: For a given SLP, if the entries of the random technological matrix A are given by $a_{ij} = a_{L_{ij}}(1+A(\gamma))$ and vectors C and R are constants where:

$a_{L_{ij}}$ is the lower bound of random variable a_{ij}

$$\gamma = \int_0^{a(\gamma)} f(a)da, \quad 0 \le \gamma \le 1,$$

Then the composition of the optimal basis X_B^* is insensitive to the random variations of A.

<u>Proof</u>: For C and r being fixed the composition of X_B^* stays the same as long as the unit vector of X_B^* is unchanged for any variations of random matrix A. For $\gamma=0$, $X_B^* = B_L^{-1}r$ and for $0<\gamma\le 1$ $X_B^* = (1+a(\gamma))B_L^{-1}r$ and as in Lemma 4.3.1 it can easily be shown that $U_{X_B^*}^0 = U_{X_B^*}^\gamma$.

<u>Theorem 4.3.1</u>: Given a SLP if

$$C_i = C_{L_i}(1+C(\alpha))$$
$$R_i = R_{L_i}(1+R(\beta))$$
$$a_{ij} = a_{L_{ij}}(1+a(\gamma))$$
$$\alpha = \int_0^{C(\alpha)} f(c)dc, \quad 0 \le \alpha \le 1$$
$$\beta = \int^{R(\beta)} f(r)dr, \quad 0 \le \beta \le 1$$
$$\gamma = \int_0^{a(\gamma)} f(a)da, \quad 0 \le \gamma \le 1$$

Then the composition of the optimal basis X_B^* is insensitive to the random variations of C, R, and A.

The proof follows from Lemmas 4.3.1, 4.3.2, and 4.3.3. The significance of balanced stochastic linear program lies in the fact that the efficient simplex algorithm can be utilized to identify the optimal basis. This is achieved by replacing the random variables by their equal percentile (e.g., mean, lower bound, upper bound) and solving the problem as a deterministic LP problem. Upon the identification of the optimal basis, the statistical properties of the optimal value can be determined. As mentioned in section 4.2 a solution of a stochastic linear program has associated with it a probability of being feasible and optimal. It is our conjecture that balanced stochastic linear programs, when solved, will yield the optimum basis with the highest probability of being feasible and optimal. This conjecture needs to be proven mathematically; however, the results of Example 4-1 (which can be considered as a dependent R-balanced model with $R_1 = 2(1+R(\beta))$; $R_2 = 3(1+R(\beta))$, and $R \sim U(0,2)$ supports our surmise for the special class of dependent balanced SLP. From Figure 4-1 it can be observed that point A (the optimal basis for $\beta=0$) and point B (the optimal basis for $\beta=1$) lie on a straight line which goes through the origin, thus a vector. The elements of this optimum basis vector are X_1 and X_2, and under the assumption of independence of R_1 and R_2, Table 4-2 shows that the optimum basis vector (X_1, X_2)

has the highest probability of being feasible and optimal. Appendix A shows the probability of possible bases of Example 4-1 for different ranges of R_1 and R_2 which again supports our conjecture.

5. APPLICATION OF THE MELLIN TRANSFORM IN STOCHASTIC LINEAR PROGRAMMING

The application of the simplex algorithm to a linear programming problem requires the multiplication and division of real fixed numbers. However, if these real numbers are subject to random variations the applicability of the standard simplex algorithm becomes questionable. In Chapter 3 we demonstrated that the Mellin transform is a powerful tool in studying the products and quotients of positive independent random variables. Therefore, it is natural to observe that the Mellin transform could be used in conjunction with the simplex algorithm to solve certain classes of stochastic linear programs. In this chapter we attempt to link the ideas presented in Chapters 2, 3, and 4, and hopefully present another view of solving certain classes of stochastic linear programs.

5.1. The Mellin Transform and the Simplex Algorithm

Consider the following stochastic linear programming problem:

$$\max Z = \sum_{j=1}^{m} C_j X_j \tag{5.1-1}$$

$$\text{s.t.} \quad \sum_{j=1}^{n} a_{ij} X_j = R_i, \quad (i = 1,m) \tag{5.1-2}$$

Assume C_j, R_i and a_{ij} $(i = 1,m;\ j = 1,n)$ are positive independent random variables with probability density functions $f_j(c)$, $f_i(r)$, and $f_{ij}(a)$, $(i = 1,m;\ j = 1,n)$, respectively. Assuming the Mellin transforms of $f_j(c)$, $f_i(r)$, and $f_{ij}(a)$ exist, the Equations 5.1-1 and 5.1-2 can be written as:

$$\max Z(s) = \sum_{j=1}^{n} F_{f_j}(C|s) \cdot X_j(s) \tag{5.1-3}$$

$$\text{s.t.} \quad \sum_{j=1}^{n} F_{f_{ij}}(a|s) \cdot X_j(s) = F_{f_i}(r|s), \quad (i = 1,m) \tag{5.1-4}$$

where:

$F_{f_i}(C|s)$ is the Mellin transform of $f_i(c)$.

$F_{f_{ij}}(a|s)$ is the Mellin transform of $f_{ij}(a)$.

$X_j(s)$ is an expression of the Mellin transform of the probability density function of ijth element of matrix B and the Mellin transform of probability density function of R_i, $(i = 1,m;\ j = 1,m)$.

$Z(s)$ is the summation of the product of $F_{f_j}(C|s)$ and $X_j(s)$, $(j = 1,m)$.

It should be noted that $X_j(s)$ and $Z(s)$ may not be the Mellin transforms of the distributions of decision variable X_j and objective value Z. The reason is the fact that the Mellin transform of the probability density function of sums of random variables is not equal to the sum of the Mellin transform of distribution of each random variable. This fact is shown by property 11 of the Mellin transform (see

Chapter 3). Equation 3.2-8, of Chapter 3, indicates that $E[Z]$ and $E[X_j]$ can directly be calculated from $Z(s)$ and $X_j(s)$ simply by evaluating the expression at $s=2$. From Equations 3.2-8 and 3.2-9 it is evident that evaluating $Z(s)$ or $X_j(s)$ at $s=3$ obtains a value bounded from above by the second moment of Z or X_j, and from below by the variance of Z or X_j, respectively.

The linear program defined by Equations 5.1-3 and 5.1-4 can be viewed as a "wait-and-see" type of stochastic linear programming problem, since for a value of $s>1$ an observation of random set (A,R,C) is made. In particular, if we assume that the above linear program is a "balanced" stochastic linear program, then the convexity of the objective function and the feasible region for an observation of random set (A, R, c) is assured.

In order to apply the simplex algorithm to the linear program defined by Equations 5.1-3 and 5.1-4 (from now on referred to as Mellin simplex algorithm), we need to restate the minimum ratio rule in the context of the Mellin transform. The minimum ratio rule for Mellin simplex algorithm can be stated as follows:

$$\underset{1\le i\le m}{\text{minimum}} \left\{ \frac{R_i(s)\big|_{s>1}}{P_{ik}(s)\big|_{s>1}};\quad P_{ik}(s)\big|_{s>1} > 0 \right\} \tag{5.1-5}$$

where:

$R_i(s)$ is the current Mellin transform expression of the right hand side. (Note, $R_i(s)$ is not necessarily the Mellin transform of the distribution of the RHS).

$P_{ik}(s)$ is the current Mellin transform expression of the ith element under the entering variable X_k. (Note, $P_{ik}(s)$ is not necessarily the Mellin transform of the distribution of the ikth element).

Let us demonstrate the Mellin simplex algorithm by solving Example 5-1, which can be viewed as an independent R-balanced stochastic linear program.

<u>Example 5-1</u>:

$$\max \quad Z = 2X_1 + X_2$$
$$\text{s.t.} \quad X_1 + X_2 + X_3 = R_1$$
$$3X_1 + X_2 + X_4 = R_2$$
$$X_1,\ X_2,\ X_3,\ X_4 \geq 0$$

where:

$$R_1 \sim U(2,6);\quad F(R_1|s) = \frac{6^s - 2^s}{4s}$$

$$R_2 \sim U(3,9);\quad F(R_2|s) = \frac{9^s - 3^s}{6s}$$

Transforming the above problem to the form of Equations 5.1-3 and 5.1-4 we obtain:

$$\max Z(s) = 2^{s-1} \cdot X_1(s) + X_2(s)$$

$$\text{s.t.} \quad X_1(s) + X_2(s) + X_3(s) = \frac{6^s - 2^s}{4s}$$

$$3^{s-1} \cdot X_1(s) + X_2(s) + X_4(s) = \frac{9^s - 3^s}{6s}$$

$$X_i(s)\Big|_{s>1} \geq 0 \quad i = 1,2,3,4 .$$

Representing in tableau form we have:

Tableau 1:

	$X_1(s)$	$X_2(s)$	$X_3(s)$	$X_4(s)$	RHS
$X_3(s)$	1^{s-1}	1^{s-1}	1	0	$\frac{6^s-2^s}{4s}$
$X_4(s)$	3^{s-1}	1^{s-1}	0	1	$\frac{9^s-3^s}{6s}$
$\bar{C}$-row	2^{s-1}	1^{s-1}	0	0	0

Use s=2 (mean) as a decision rule to change the basis. Then in Tableau 1 $X_1(s)$ is a candidate to enter the basis. The leaving variable is determined by Equation 5.1-5

$$\text{Minimum } \left\{ \frac{\left.\frac{6^s-2^s}{4s}\right|_{s=2}}{\left.1^{s-1}\right|_{s=2}} ; \ \frac{\left.\frac{9^s-3^s}{6s}\right|_{s=2}}{\left.3^{s-1}\right|_{s=2}} \right\} = \text{minimum } \{4;2\}$$

Thus $X_4(s)$ is a candidate to leave the basis. Multiplying the second row by 3^{1-s} and performing the necessary row operations obtains the second tableau.

Tableau 2:

	$X_1(s)$	$X_2(s)$	$X_3(s)$	$X_4(s)$	RHS
$X_3(s)$	0	$1-3^{1-s}$	1	-3^{1-s}	$\frac{6^s-2^s}{4s} - \frac{9^s-3^s}{6s} \cdot 3^{1-s}$
$X_1(s)$	1	3^{1-s}	0	3^{1-s}	
$\bar{C}$-row	0	$1-2^{s-1}.3^{1-s}$	0	$-2^{s-1}.3^{1-s}$	$-2^{s-1}.\frac{9^s-3^s}{6s} \cdot 3^{1-s}$

The coefficient of $X_2(s)$ in $\bar{C}$-row evaluated at s=2 is positive, thus $X_2(s)$ is the candidate to enter the basis. The leaving variable is determined by the minimum of the following ratios:

$$\frac{\left(\frac{6^s-s^2}{4s} - \frac{9^s-3^s}{6s}\right)\Big|_{s=2}}{(1-3^{1-s})\Big|_{s=2}} = 3$$

and

$$\frac{\left(\frac{9^s-3^s}{6s} \cdot 3^{1-s}\right)\Big|_{s=2}}{(3^{1-s})\Big|_{s=2}} = 6$$

Therefore, $X_3(s)$ will leave the basis. Following row operations Tableau 3 is obtained.

Tableau 3:

	$X_1(s)$	$X_2(s)$	$X_3(s)$	$X_4(s)$
$X_2(s)$	0	1	$\frac{1}{1-3^{1-s}}$	$\frac{-3^{1-s}}{1-3^{1-s}}$
$X_1(s)$	1	0	$-\frac{3^{1-s}}{1-3^{1-s}}$	$\frac{3^{1-s}}{1-3^{1-s}}$
$\bar{C}$-row	0	0	$\frac{-1+2^{s-1}.3^{1-s}}{1-3^{1-s}}$	$\frac{3^{1-s}(1-2^{s-1})}{1-3^{1-s}}$

Tableau 3 (Continued):

	RHS
$X_2(s)$	$\frac{R_1-R_2.3^{1-s}}{1-3^{1-s}}$
$X_1(s)$	$\frac{3^{1-s}(R_2-R_1)}{1-3^{1-s}}$
$\bar{C}$-row	$\frac{R_1(2^{s-1}.3^{s-1}-1)+R_2.3^{1-s}(1-2^{s-1})}{1-3^{1-s}}$

Where

$$R_1 = \frac{6^s-2^s}{4s} \quad \text{and} \quad R_2 = \frac{9^s-3^s}{6s}$$

Since the elements of $\overline{C}$-row in Tableau 3 are all nonpositive, we have reached an optimal solution. The optimal value of X_2, X_1 and negative of Z are shown in the RHS column of Tableau 3. It should be noted that since the above example is a balanced stochastic linear program the choice of s will not change the composition of the optimum basis.

Perhaps the only situation that one might be willing to use the Mellin simplex algorithm is the case when each element of random set (A, R, C) is:

a) some power of a known random variable;

b) a quotient of known random variables;

c) a product of some known random variables.

For example, if the profit coefficient of decision variable X_1 is C_1^{α} (α is real) where the p.d.f. of C_1 is known to be $f_1(C)$, then by using property 4 of the Mellin transform, in Equation 5.1-3 $F_{f_1}(C|s)$ becomes $F_{f_1}(C|\alpha s-\alpha+1)$. Disregarding the above situations, the computational effort needed to solve a given problem by the Mellin simplex algorithm does not justify its usage.

5.2. SFG Solution of Stochastic Linear Program

In Chapter 2 we showed how a deterministic linear program can be solved using signal flow graph (SFG) procedures. In this section the method developed in Chapter 2 is utilized in conjunction with the idea discussed in Section 5.1 to give another view of solving certain class of stochastic linear programs.

Consider the linear program of Equations 5.1-3 and 5.1-4 stated below

$$\max Z(s) = \sum_{j=1}^{n} F_{f_j}(C|s) \cdot X_j(s) \tag{5.2-1}$$

$$\text{s.t.} \quad \sum_{j=1}^{n} F_{f_{ij}}(a|s) \cdot X_j(s) = F_{f_i}(R|s); \quad (i = 1,m) \tag{5.2-2}$$

Putting the above SLP model in SFG standard form, that is determining the dependent (basic) variables, and assuming random variables a_{ij} do not consist of addition or subtraction of random variables we obtain.

$$\max Z(s) = \sum_{j=1}^{n} F_{f_j}(C|s) X_j(s) \tag{5.2-3}$$

$$\text{s.t.} \quad X_i(s) = F_{f_{ii}}(a|-s+2) \left[\sum_{\substack{j=1 \\ j \neq i}}^{n} F_{f_{ij}}(a|s) . X_j(s) + F_{f_i}(R|s) \right]; \quad (i = 1,m) \tag{5.2-4}$$

Assuming the decision rule to change the basis has been specified, then by using the method discussed in Section 2.4 the optimum solution to the above model can be determined. The solution procedure is demonstrated by solving Example 4-1.

Example 5-2:

$$\max Z(s) = 2^{s-1} \cdot X_1(s) + X_2(s)$$

$$\text{s.t.} \quad X_1(s) + X_2(s) + X_3(s) = R_1$$

$$3^{s-1}X_1(s) + X_2(x) + X_4(s) = R_2$$

$$X_i(s)\Big|_{s>1} \geq 0 \qquad i = 1,2,3,4 .$$

where

$$R_1 = \frac{6^s - 2^s}{4s} \quad \text{and} \quad R_2 = \frac{9^s - 3^s}{6s}$$

Selecting $X_3(s)$ and $X_1(s)$ as the dependent variable we obtain

$$\max Z(s) = 2^{s-1} \cdot X_1(s) + X_2(s)$$

$$\text{s.t.} \quad X_3(s) = -X_1(s) - X_2(s) + R_1$$

$$X_1(s) = -3^{1-s} \cdot X_2(s) - 3^{1-s} \cdot X_4(s) + 3^{1-s}R_2$$

Using the graphical symbols of Table 2-1, the SFG representation of the above model is shown in Figure 5-1.

$$X_1(s) = (T_{R_1 \to X_1})R_1 + (T_{R_2 \to X_1})R_2 = 3^{1-s}R_2$$

$$X_2(s) = (T_{R_1 \to X_2})R_1 + (T_{R_2 \to X_2})R_2 = 0$$

$$X_3(s) = (T_{R_1 \to X_3})R_1 + (T_{R_2 \to X_3})R_2 = R_1 - 3^{1-s}R_2$$

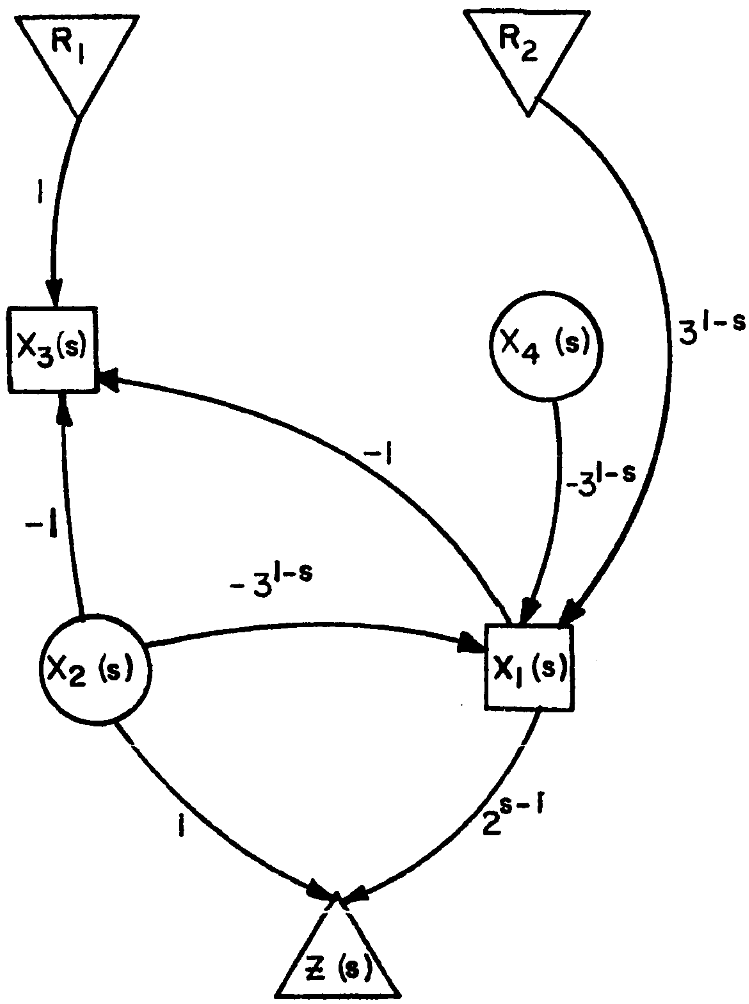


Figure 5-1. SFG representation of Stage 1 of Example 5-2

$$X_4(s) = (T_{R_1 \to X_3})R_1 + (T_{R_2 \to X_3})R_2 = 0$$

$$Z(s) = (T_{R_1 \to Z})R_1 + (T_{R_2 \to Z})R_2 = 3^{1-s}.2^{s-1}R_2$$

Determination of entering and leaving variables: assume our decision rule is based on s=2 (mean).

$$T_{X_4 \to Z}\Big|_{s=2} = -3^{1-s}.2^{s-1}\Big|_{s=2} = -\frac{2}{3}$$

$$T_{X_2 \to Z}\Big|_{s=2} = 1-3^{1-s}.2^{s-1} = \frac{1}{3}$$

Since the maximum $[T_{X_4 \to Z}, T_{X_2 \to Z}] = \frac{1}{2}$, X_2 is a candidate to enter the basis.

$$T_{X_2 \to X_1}\Big|_{s=2} = -3^{1-s}\Big|_{s=2} = -\frac{1}{3}$$

$$X_1(s)\Big|_{s=2} = 3^{1-s}.\ \frac{9^s-3^s}{6s}\Big|_{s=2} = 2$$

$$R_1 = \frac{2}{-\frac{1}{3}} = -6$$

$$T_{X_2 \to X_3}\Big|_{s=2} = -1 + 3^{1-s}\Big|_{s=2} = \frac{-2}{3}$$

$$X_3(s)\Big|_{s=2} = \left(\frac{6^s-2^s}{4s} - 3^{1-s}.\ \frac{9^s-3^s}{6s}\right)\Big|_{s=2} = 2$$

$$R_3 = \frac{2}{-\frac{2}{3}} = -3$$

Note that:

Maximum $[R_1, R_3] = -3$, thus X_3 is a candidate to leave the basis.

Stage 2:

$$\max Z(s) = 2^{s-1} \cdot X_1(s) + X_2(s)$$

$$\text{s.t.} \quad X_2(s) = -X_1(s) - X_3(s) + R_1$$

$$X_1(s) = -3^{1-3} \cdot X_2(s) - 3^{1-s} \cdot X_4(s) + 3^{1-s} R_2$$

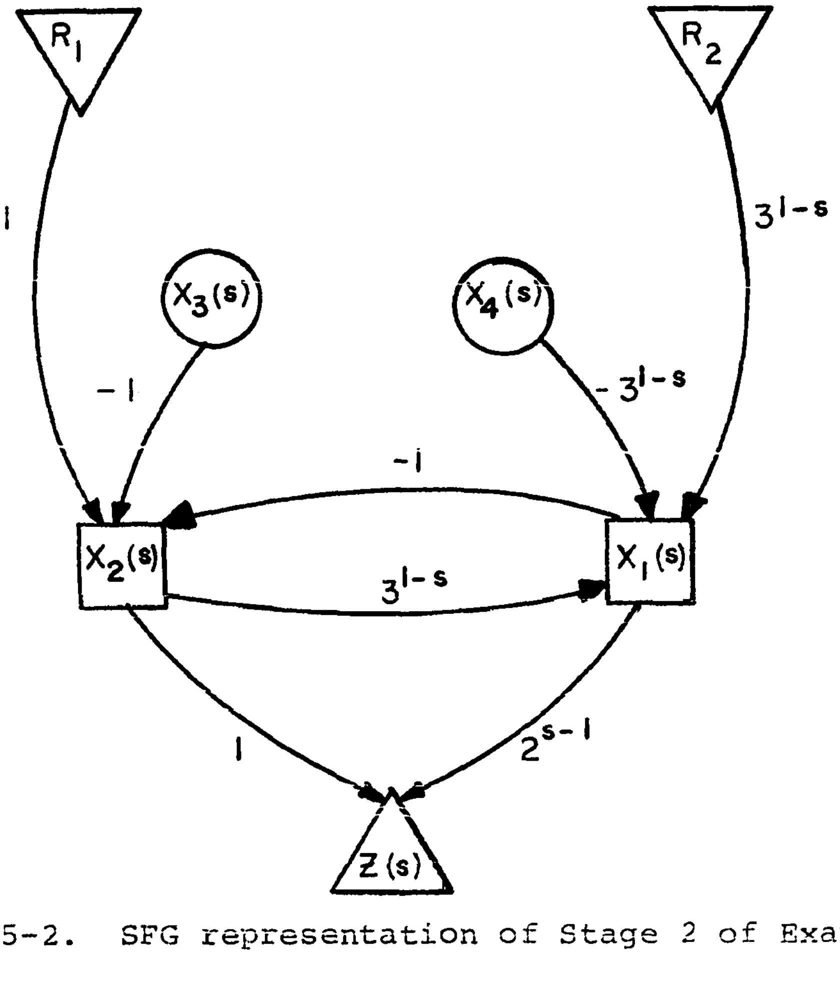


Figure 5-2. SFG representation of Stage 2 of Example 5-2

$$X_1(s) = (T_{R_1 \to X_1})R_1 + (T_{R_2 \to X_1})R_2 = \frac{-3^{1-s}R_1+3^{1-s}R_2}{1-3^{1-s}}$$

$$X_2(s) = (T_{R_1 \to X_2})R_1 + (T_{R_2 \to X_2})R_2 = \frac{R_1-3^{1-s}R_2}{1-3^{1-s}}$$

$$X_3(s) = (T_{R_1 \to X_3})R_1 + (T_{R_2 \to X_3})R_2 = 0$$

$$X_4(s) = (T_{R_1 \to X_4})R_1 + (T_{R_2 \to X_4})R_2 = 0$$

$$Z(s) = (T_{R_1 \to Z})R_1 + (T_{R_2 \to Z})R_2$$

$$= \frac{(1-3^{1-s}.2^{s-1})R_1 + (3^{1-s}.2^{s-1}-3^{s-1})R_2}{1-3^{1-s}}$$

Since $T_{X_3 \to Z}\Big|_{s=2} = -\frac{1}{2}$, and $T_{X_4 \to Z}\Big|_{s=2} = -\frac{1}{2}$ are both nonpositive the current solution, given our decision criterion (s=2), is optimal. Using property 11 of the Mellin transform the mean and variance of the optimal values can be determined.

The situation that the random variables a_{ij}, C_j, and/or R_i ($i = 1,m$; $j = 1,n$) follow a discrete distribution can be handled with a small modification in the SFG representation. For example, suppose R_1 and R_2 of Example 5-1 follow the following distributions:

$$R_1 = \begin{cases} R_1' & \text{with probability } p_1 \\ R_1'' & \text{with probability } 1-p_1 \end{cases}$$

and

$$R_2 = \begin{cases} R_2' & \text{with probability } p_2 \\ R_2'' & \text{with probability } 1-p_2 \end{cases}$$

where R_1', R_1'', R_2', and R_2'' are some known Mellin transforms. p_1 and p_2 are constants [0,1]. Then the final SFG of Example 5-1 shown in Figure 5-2 can be redrawn as Figure 5-3.

Depending on the R_1', R_1'', R_2', R_2'', and the decision rule used, Figure 5-3 may not be the optimum SFG. Suppose for the sake of argument that Figure 5-3 represents an optimal SFG, then the Mellin transform expressions of the optimal solutions are:

$$X_1(s) = \frac{3^{1-s}[p_2R_2'+(1-p_2)R_2''] - 3^{1-s}[p_1R_1'+(1-p_1)R_1'']}{1-3^{1-s}}$$

$$X_2(s) = \frac{[p_1R_1'+(1-p_1)R_1''] - 3^{1-s}[p_2R_2'+(1-p_2)R_2'']}{1-3^{1-s}}$$

$$X_3(s) = 0$$

$$X_4(s) = 0$$

$$Z(s) = \frac{(1-3^{1-s}.2^{s-1})[p_1R_1'+(1-p_1)R_1''] + (3^{1-s}.2^{s-1}-3^{s-1})[p_2R_2'+(1-p_2)R_2'']}{1-3^{1-s}}$$

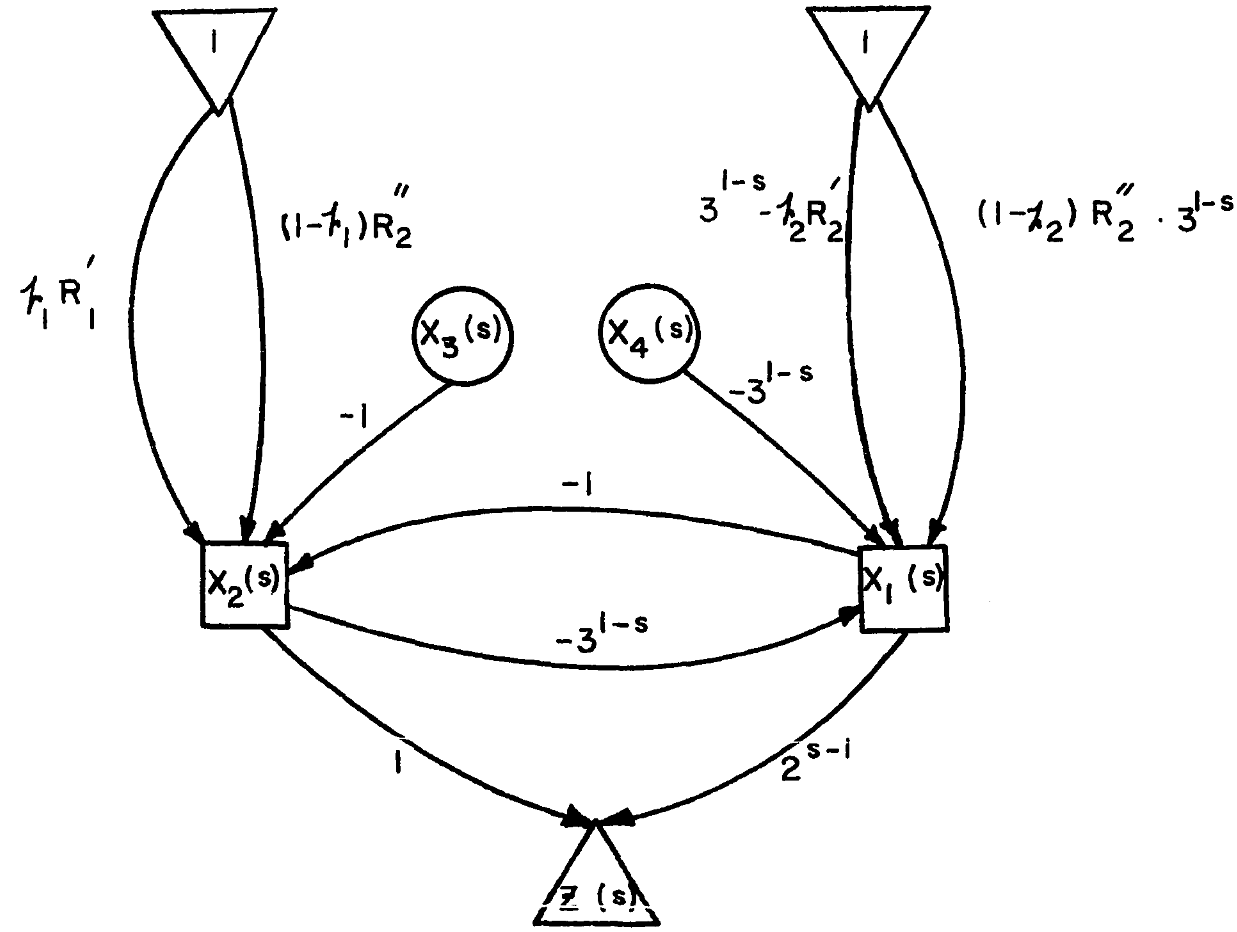


Figure 5-3. SFG representation of Stage 2 of Example 5-2 with discrete random resources

5.3. Postoptimality Analysis and Solution Methods Evaluation

Postoptimality analysis in a stochastic environment by SFG, assuming the decision rule is specified, is similar to the deterministic case with some minor modifications. For example, to determine the variations of profit coefficient c_j, in the final SFG we simply replace $F_{f_j}(c|s)$ by $F_{f_j}(c|s) + \Delta_j^{s-1}$ and proceed in a manner similar to the Section 2.5.1 for some s>1. For finding the range of r_i or a_{ij} we replace $F_{f_i}(r|s)$ by $F_{f_i}(r|s) + \Delta_i^{s-1}$ or $F_{f_{ij}}(a|s)$ by $F_{f_{ij}}(a|s) + \Delta_{ij}^{s-1}$ and follow the procedure of Section 2.5 for some s>1.

In this chapter we presented different views of solution methods of a certain class of stochastic linear programs. The practicality of these procedures are questionable, with the exception of situations mentioned at the end of Section 5.1. For a small problem, suitable to the procedures explained in this chapter, it appears that the SFG method may be easier computationally than the Mellin simplex algorithm.

6. CONCLUSIONS AND RECOMMENDATIONS

A procedure to determine the inverse of a matrix based on the concept of signal flow graph has been presented. It has been mentioned that this procedure can be an attractive method in the situations where the matrix has a high degree of sparsity. A formal treatment of solving the standard LP problem using the Signal Flow Graph procedure in conjunction with the simplex algorithm has been presented. No computational efficiency is noted for this procedure except in the situation where the technological matrix is highly sparse. Chapter 3 presented a formal discussion of applications of the Mellin transform in statistics. Some extensions of the present application are offered. In particular: a) the Mellin transform of the cumulative distribution was derived, b) the theoretical approach of finding the Mellin transform of a truncated cumulative distribution was discussed, c) the idea of semi-variance in continuous form was stated, and a theoretical way of determining the semi-variance from the Mellin transform of the probability density function was discussed, d) a way of calculating mean and variance of summation of independent, and also perfectly correlated random variables from their Mellin transforms was presented.

In Chapter 4 some of the complexities of determining the distribution of the optimal solution for the models with stochastic profit and resource vectors were discussed. While Section 4.2 gives an optimality and feasibility condition for certain stochastic linear programming models. In order to determine the distribution of the optimal solution one must calculate the probability of feasibility and optimality of all possible bases, albeit impractical for most real world problems. Also, a new class of stochastic linear programming problem, called balanced SLP, was introduced. This class of SLP has the interesting property that the efficient simplex algorithm may be used to determine the composition of the optimum basis. This is achieved by replacing the random variables with an equal percentile observation (such as mean) of the random variable, and to solve the deterministic LP problem.

In Chapter 5 it was shown how Mellin transform can be used in a simplex tableau form or with SFG to solve certain classes of stochastic linear programs. It has been noted that the computational effort needed to solve a given SLP by the methods of Chapter 5 does not justify its usage. Perhaps the most important area which needs further research is the determination of a (or some) practical decision rule(s) for changing the basis and its related optimality criterion for different classes of stochastic linear programming models.

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9. APPENDIX A: PROBABILITIES OF POSSIBLE BASES OF EXAMPLE 4-1 FOR DIFFERENT RANGES OF R_1 AND R_2

Table 9.1. The results of Example 4-1 with the RHS of: $R_1 \sim U(0,2)$; $R_2 \sim U(3,9)$

ℓ	Basis	Prob. that the basis is feasible	Prob. that the basis is optimal	Value of the objective function	Prob. that the basis is feasible optimal
1	X_1 & X_2	0.125	1.0	$\frac{1}{2}(R_1 + R_2)$	0.125
2	X_1 & X_3	0.125	0.0	$2/3\ R_2$	0.0
3	X_1 & X_4	0.875	1.0	$2R_1$	0.875
4	X_2 & X_3	0.0	1.0	R_2	0.0
5	X_2 & X_4	1.0	0.0	R_1	0.0
6	X_3 & X_4	1.0	0.0	0	0.0

Table 9.2. The results of Example 4-1 with the RHS of: $R_1 \sim U(3,9)$; $R_2 \sim U(1,3)$

ℓ	Basis	Prob. that the basis is feasible	Prob. that the basis is optimal	Value of the objective function	Prob. that the basis is feasible optimal
1	X_1 & X_2	0.0	1.0	$\frac{1}{2}(R_1 + R_2)$	0.0
2	X_1 & X_3	1.0	0.0	$2/3\ R_2$	0.0
3	X_1 & X_4	0.0	1.0	$2R_1$	0.0
4	X_2 & X_3	1.0	1.0	R_2	1.0
5	X_2 & X_4	0.0	0.0	R_1	0.0
6	X_3 & X_4	1.0	0.0	0	0.0

Table 9.3. The results of Example 4-1 with the RHS of: $R_1 \sim U(0,10)$; $R_2 \sim U(0,2)$

ℓ	Basis	Prob. that the basis is feasible	Prob. that the basis is optimal	Value of the objective function	Prob. that the basis is feasible optimal
1	X_1 & X_2	0.00667	1.0	$\frac{1}{2}(R_1 + R_2)$	0.00667
2	X_1 & X_3	0.96667	0.0	2/3 R_2	0
3	X_1 & X_4	0.03333	1.0	$2R_1$	0.03333
4	X_2 & X_3	0.9	1.0	R_2	0.9
5	X_2 & X_4	0.1	0.0	R_1	0.0
6	X_3 & X_4	1.0	0.0	0	0.0

10. APPENDIX B: MELLIN TRANSFORMS FOR SELECTED PROBABILITY DENSITY FUNCTIONS

Table 10.1. Mellin transforms for selected p.d.f.s.

p.d.f., f(x)	$F_f(x\|s)$
a, constant	a^{s-1}
$\frac{1}{b-a}$, $a \le x \le b$	$\frac{b^s - a^s}{s(b-a)}$
e^{-x}, $x>0$	$\Gamma(s)$
ae^{-ax}, $x>0$	$(\frac{1}{a})^{s-1}\Gamma(s)$
$\frac{x^a e^{-x}}{\Gamma(a+1)}$, $0<x<\infty$, $a>-1$	$\frac{\Gamma(a+s)}{\Gamma(a+1)}$
$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$ $0<x<1;\ \alpha>0;\ \beta>0$	$\frac{\Gamma(\alpha+\beta)\Gamma(\alpha+s-1)}{\Gamma(\alpha+\beta+s-1)\Gamma(a)}$
$\frac{2}{a^2}x \quad a>0$	$\frac{2}{a^2}\frac{a^{s+1}}{s+1}$
$\begin{cases} \frac{2(x-a)}{(c-a)(b-a)}, & 0 \le a \le x \le b \\ \frac{-2(x-b)}{(c-a)(c-b)} + \frac{2}{c-a} & b \le x \le c \end{cases}$	$\frac{2ab+2bc-4ac}{(b-a)(c-a)(c-b)} + \frac{2(b^{s+1}-a^{s+1})}{(b-a)(c-a)(s+1)} - \frac{2(c^{s+1}-b^{s+1})}{(c-a)(c-b)(s+1)}$
$\frac{1}{\pi(1+x^2)}$ $-\infty<x<\infty$	$\frac{1-(-1)^s}{2}\ \mathrm{Cosec}(\frac{\pi s}{2})$
$\frac{1}{2}e^{-\|x\|}$ $-\infty<x<\infty$	$\frac{1-(-1)^s}{2}\ \Gamma(s)$

11. APPENDIX C: SIGNAL FLOW GRAPHS

A flow graph is a topological portrayal of a system of linear algebraic equations. S. J. Mason (1953, 1956) recognized the mathematical structure of the flowgraphs and formulated precise rules for the graphical manipulation of a set of linear algebraic equations. Since the original applications of these concepts by Mason (1953) were in the area of electronics, he coined the name "signal flowgraphs" (SFG). As the name implies, SFG depicts the flow of signals from one point of the system to another. Irrespective of the original content many systems can be modeled as a set of linear algebraic equations to which the methodology of SFG can directly be applied.

11.1. Basic Concepts and Terminology

Suppose a linear system can be described mathematically by the following set of linear algebraic equations.

$$\sum_{j=1}^{n} a_{ij}x_j + b_i = y_i \qquad i = 1,\ldots m$$

where

y_i = dependent variable

b_i = resource or initial condition

x_j = independent variable

a_{ij} = relationship between y_i and x_j.

11.1.1. Terminology

a) Dependent or independent variables are depicted by circles called nodes.

b) When relationships exist between nodes, then branches are used to represent such relationships.

A branch has the following properties:

i) It is a directed line joining two nodes.

ii) It has a magnitude called transmittance or branch-gain which is determined by the relationship between two nodes (a_{ij}).

iii) It has a direction which is indicated by an arrow from independent to dependent variable.

c) When no branches emanate from a node, this node is called a sink node.

d) When no branches have their arrow pointing toward a particular node, this node is called a source node.

e) A chain node is a node which has only one incoming as well as one emanating branch.

f) A loop is a collection of branches which are connected only by chain nodes (i.e., the source and the sink nodes are the same).

g) A self-loop is a loop which contains only a single chain node.

h) A path is a series of branches which join some nodes and it does not pass through a node more than once.

i) A forward path is a path which starts from source node and ends at the sink node.

j) A collection of loops is said to be nontouching if no two of the loops have a node in common.

Example 11.1: Consider the following set of linear algebraic equations:

$$x_1 = x_2 - 4x_3$$
$$x_2 = x_1 + 3x_2 + 2x_3$$
$$x_3 = x_2 - x_4$$

x_1 is the only dependent variable in the first equation. The SFG representation of the equation $x_1 = x_2 - 4x_3$ is:

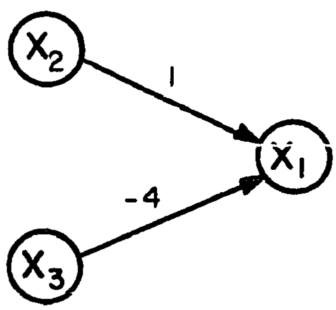

Figure 11-1. SFG representation of equation: $X_1 = X_2 - 4X_3$

The summation is indicated by the converging arrows. It should be noted that if x_1 is treated as a dependent variable, then the corresponding SFG is unique.

The SFG of the above set of equations is shown as follows:

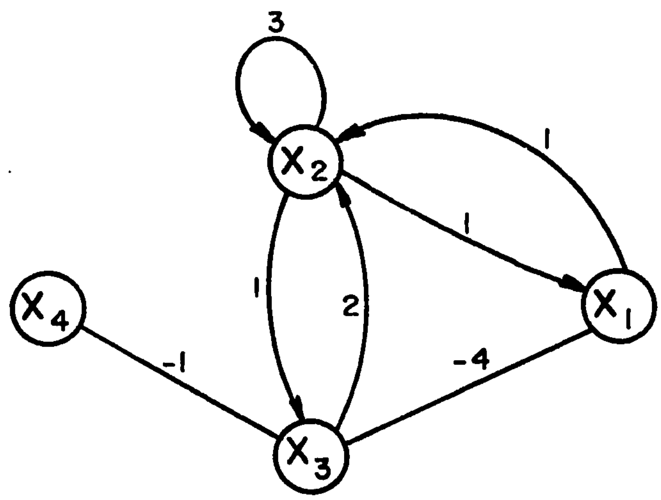

Figure 11-2. SFG representation of equations of Example C-1

11.1.2. Path inversion

The process of interchanging the dependent variable and an independent variable is called path inversion. This process can be accomplished by rewriting the equation in terms of the new dependent variable and redrawing the graph or graphically by implementing the following steps.

1) Change the direction of the arrow between the old and the new dependent variable and invert the branch transmittance.

2) Divert the branches from the other independent variables and resources or initial condition to the new dependent variable. Divide their transmittance by the negative of transmittance of the branch connecting the new to the old dependent variable.

Example 11-2: Consider the equation

$$x_4 = ax_1 + bx_2 + cx_3 + b_1$$

in which x_4 is the dependent variable and x_1, x_2, and x_3 are independent variables. Assuming x_1 is the new dependent variable the equation can be rewritten as follows:

$$x_1 = - \frac{b}{a} x_2 - \frac{c}{a} x_3 + \frac{1}{2} x_4 - \frac{1}{a} b_1$$

The graphical representation of these equations are shown below:

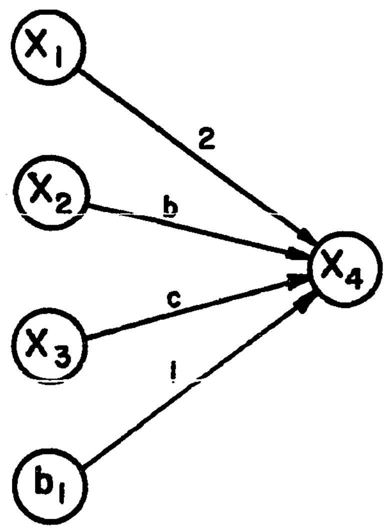

Figure 11-3a. SFG representation of equation: $X_4 = ax_1+bx_2+cx_3+b_1$

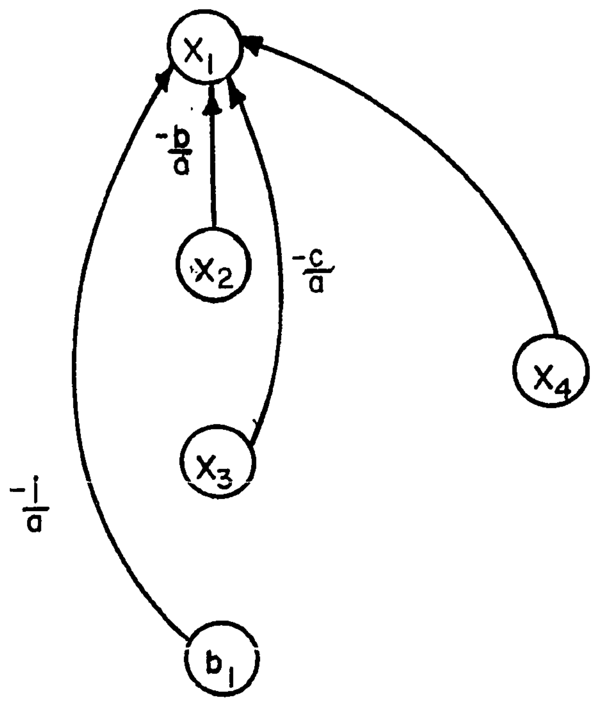

Figure 11-3b. SFG representation of equation:

$$X_1 = \frac{b}{a}X_2 - \frac{c}{a}X_3 + \frac{1}{a}X_4 - \frac{b_1}{a}$$

11.1.3. Method of solution of SFG

S. J. Mason (1956) developed a general rule for finding the gain of a signal flow graph (SFG) for a linear system described by

$$\sum_{j=1}^{n} a_{ij}x_j = y_i \qquad i = 1,\ldots,m$$

The gain $T_{j\to i} = \frac{y_i}{x_j}$ represents the linear dependence between a dependent variable y_i and an independent variable x_j. $T_{j\to i}$ is the transmittance or gain from node x_j to node y_i.

The gain $T_{j\to i}$ can be calculated from the corresponding signal flow graph by means of Mason's formula

$$T_{j\to i} = \frac{\Sigma_k\ L_{jik}\Delta_{jik}}{\Delta}$$

where:

L_{jik} = k^{th} forward path from variable x_j to y_i

Δ = determinant of the graph

$\Delta = 1 - \Sigma L_i + \Sigma L_i L_j - \Sigma L_i L_j L_k + \ldots$

ΣL_i = summation of the gain of all loops in the graph

$\Sigma L_i L_j$ = summation of the gain of all pairs of non-touching loops in the graph

$\Sigma L_i L_j L_k$ = summation of the gain of all triplet of nontouching loops in the graph

Δ_{jik} = cofactor of the path L_{jik}; the cofactor Δ_{jik} is the determinant of the system with path L_{jik} removed.

C. S. Lorens (1964), S. J. Mason (1953), and Y. Chow and Cassignol (1962) give proof of the Mason's Formula.

Example 11-3: Consider the signal flow graph shown below:

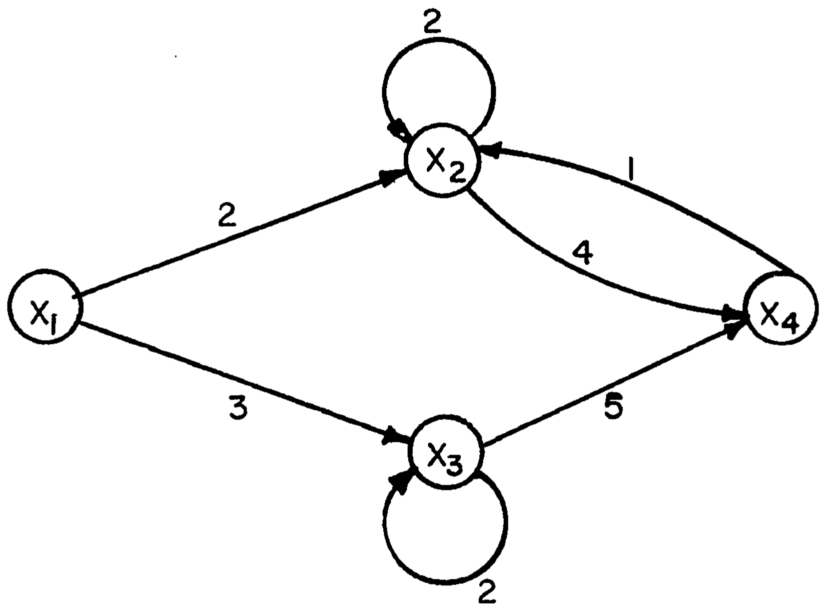

Figure 11-4. SFG representation of Example C-3

Determine the gain between node x_1 and x_4.

Solution: Applying Mason's Formula we find:

$L_{141} = (2)(4) = 8$

$L_{142} = (3)(5) = 15$

$\Delta = 1 - [2+(4)(1)+(+2)] + [(2)(+2)+(2)(4)(1)]$

$= 1 - 8 + 12 = 5$

$\Delta_{141} = 1 - 2 = -1$

$\Delta_{142} = 1 - [+2+4] = -5$

$T_{14} = \dfrac{(8)(-1) + (15)(-5)}{5} = \dfrac{83}{5}$

Zadeh and Desoer (1963) show that the Mason's gain formula can be viewed equivalent to the Cremar's rule of solving a system of equation.